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A hierarchical model of tail dependent asset returns for assessing portfolio credit risk

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Abstract

This paper introduces a multivariate pure-jump Lévy process which allows for skewness and excess kurtosis of single asset returns and for asymptotic tail dependence in the multivariate setting. It is termed Variance Compound Gamma (VCG). The novelty of my approach is that, by applying a two-stage stochastic time change to Brownian motions, I derive a hierarchical structure with different properties of inter- and intra-sector dependence. I investigate the properties of the implied static copula families and come to the conclusion that they are ordered with respect to their parameters and that the lower-tail dependence of the intra-sector copula is increasing in the absolute values of skewness parameters. Furthermore, I show that the joint characteristic function of the VCG asset returns can be explicitly given as a nested Archimedean copula of their marginal characteristic functions. Applied to credit portfolio modelling, the framework introduced results in a more conservative tail risk assessment than a Gaussian framework with the same linear correlation structure, as I show in a simulation study. To foster the simulation efficiency, I provide an Importance Sampling algorithm for the VCG portfolio setting.

Keywords: Portfolio Credit Risk, Stochastic Time Change, Brownian Subordination, Jumps, Tail Dependence, Hierarchical Dependence Structure

JEL Classification: C46, C63, G12, G21

Non-technical summary

In this paper, I introduce a novel model of tail-dependent asset returns which can be used for the purposes of structural credit risk modelling. Similar to the copula approach proposed recently by Puzanova (2011), the Variance Compound Gamma (VCG) model presented here implies a hierarchical dependence structure with stronger dependence within pre-specified sectors than between them. The magnitude of sector-specific dependence parameters governing the tail dependence property can vary from one sector to another, allowing the model to cope with concentration risk. An advantage of the VCG framework over the aforementioned copula approach is its more general applicability, which is not limited to a static, one-period consideration of portfolio credit risk. In fact, the fundamental VCG model of asset returns can be utilised for financial modelling (pricing of financial derivatives etc.) whenever using multivariate jump-driven Lévy processes with a hierarchical dependence structure is deemed appropriate.

Allowing jumps in the sample paths of the asset returns, the VCG model overcomes the shortcomings of a Gaussian/Brownian framework, such as anticipated default time, symmetric and mesokurtic probability distribution of the underlying and linear dependence structure. The jumps occur simultaneously for asset returns that are evaluated at a common business time. A business time common to all assets in an appropriately specified sector is a stochastic process which represents the irregular flow of information that is only relevant for that particular sector. The sector-specific business times themselves are evaluated at another common random time, which represents the flow of information that is relevant for the whole market, such as changes in the overall macroeconomic conditions. This two-stage stochastic time change is a novelty of my approach. It results in the hierarchical dependence structure of the stochastic processes governing asset returns.

To assess the effect of the VCG modelling approach on the credit portfolio tail risk measures and, thus, to gauge the extent of model risk, I conduct a simulation study for credit portfolios with identical linear correlation structure but different tail dependence properties of underlying asset returns. I show that the Gaussian model underestimates the portfolio tail risk considerably if the assumption of asymptotically independent extreme asset returns is wrong. For instance, the Value at Risk and Expected Shortfall calculated at the 99.9% level for a stylised portfolio containing 100 (1,000) obligors are about 25% (50%) higher under the assumptions of the model introduced in this paper than under the Gaussian assumptions.

In view of these results, the proposed model could have implications for risk controlling and banking regulation and, on a large scale, for financial stability. Its implementation would result in a more conservative assessment of portfolio tail risk and, consequently, higher capital requirements. Therefore, the model is able to counter the systematic underestimation of credit risk in banking sector – one of the basic causes of the recent financial turmoil.

Nichttechnische Zusammenfassung

In diesem Beitrag stelle ich einen neuen Ansatz zur multivariaten Modellierung stochastischer Assetrenditen vor, deren extreme Realisationen untereinander abhängig sind (Flankenabhängigkeit). Dieser Ansatz eignet sich insbesondere zur Modellierung von Firmenwertrenditen der Kreditnehmer im Rahmen eines strukturellen Kreditportfoliomodells. Ähnlich dem Copula-Ansatz von Puzanova (2011) impliziert das hier vorgestellte Variance-Compound-Gamma-Modell (VCG) eine hierarchische Abhängigkeitsstruktur, sodass die Abhängigkeit zwischen den Firmenwertrenditen der Schuldner, die dem gleichen Sektor zugeordnet sind, stärker ist als zwischen den Firmenwertrenditen der Schuldner aus verschiedenen Sektoren. Anders als erwähnter Copula-Ansatz beschränkt sich das VCG-Modell nicht auf eine statische Darstellung eines Kreditportfolios. Vielmehr kann es immer dann in der Finanzmodellierung eingesetzt werden, wenn Anwendung multivariater Lévy-Prozesse, deren Pfade Sprünge aufweisen und die eine hierarchische Abhängigkeitsstruktur besitzen, sinnvoll erscheint.

Die Einführung von Sprüngen in die Modellierung stochastischer Renditeprozesse beseitigt solche Mängel eines Gaußschen bzw. Brownschen Modells wie antizipierte Ausfallzeiten, symmetrische und mesokurtische Wahrscheinlichkeitsverteilung der Renditen sowie ausschließlich lineare Abhängigkeiten. Die Sprünge werden eingeführt, indem stochastische Zeiten zur Evaluierung der Brownschen Renditeprozesse eingesetzt werden. Eine stochastische Zeit repräsentiert den unregelmäßigen Informationsfluss auf dem Markt. Die Flankenabhängigkeit der Assetrenditen im gleichen Sektor geht somit auf die gleichzeitig stattfindenden Sprünge in den Renditeprozessen zurück. Diese Sprünge finden genau dann statt, wenn die für diesen Sektor relevanten Informationen eintreffen. Die sektorspezifischen stochastischen Zeiten werden ihrerseits auf einer einheitlichen, marktübergreifenden stochastischen Zeit evaluiert. Die Letztere repräsentiert den unregelmäßigen Fluss solcher Informationen, die für alle Sektoren relevant sind, wie z.B. Informationen über die gesamtwirtschaftlichen Bedingungen. Die beschriebene zweistufige Zeitänderung, die in einer hierarchischen Abhängigkeitsstruktur der Assetrenditen resultiert, stellt eine Innovation in der Finanzmodellierung dar.

Um das Ausmaß des Modellrisikos abzuschätzen, führe ich eine Simulationsstudie durch. Ich betrachte stilisierte Portfolien mit einer vorgegebenen linearen Korrelationsstruktur aber unterschiedlichen Eigenschaften in Bezug auf die Flankenabhängigkeit der Risikofaktoren. Die Ergebnisse zeigen, dass ein Gaußsches Modell das Portfoliotailrisiko erheblich unterschätzt, wenn die Annahme der asymptotisch unabhängigen Flanken nicht zutrifft. So ergibt das hier vorgestellte Modell Werte des Value at Risk und Expected Shortfall (zu 99,9%), die für ein stilisiertes Portfolio aus 100 (1.000) Exposures um ca. 25% (50%) höher liegen als im Falle des Gaußschen Modells. Somit würde die Anwendung des hier vorgestellten Ansatzes zur Kreditportfoliomodellierung in einer konservativen Einschätzung unerwarteter Portfolioverluste und, damit einhergehend, in einer höheren Kapitalunterlegung der Banken resultieren. Dies könnte letztlich einen stabilisierenden Effekt auf das gesamte Finanzsystem haben.

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A hierarchical model of tail dependent asset returns for assessing portfolio credit risk[1]

1. Introduction

This paper introduces a multivariate stochastic model for logarithmic asset returns which accounts for such stylised facts as skewness and excess kurtosis of the marginal probability distributions of asset returns and tail dependence of their joint distributions. The model is derived by evaluating initially independent Brownian motions, grouped by sectors, at the sector-specific stochastic chronometers. This stochastic time represents the irregular information flow relevant for doing business in the respective sector. As the companies in different sectors may also not be entirely independent of one another, the sector-specific stochastic chronometers are themselves evaluated at an independent common stochastic chronometer that represents the flow of general information relevant for all firms in the market, such as changes in the overall macroeconomic conditions.

The specific time-change procedure described is a novelty of my approach. To the best of my knowledge, it is the first paper to utilise a two-stage stochastic time change in order to generate multidimensional Lévy processes with a *hierarchical* dependence structure. The model's hierarchical structure has the advantage of allowing for a *stronger dependence within* given economic or geographic sectors or certain sub-portfolios as compared to a *weaker dependence between* the sectors/sub-portfolios. Moreover, the magnitude of sector-specific parameters governing tail dependence may vary from one sector to another.

Both distinguishing properties of the specific multivariate model introduced in this paper – (i) the hierarchical dependence structure and (ii) the tail dependence – are highly relevant from the perspective of credit portfolio modelling. The first property is desirable because companies in the same sector usually exhibit stronger dependence. The degree of dependence between the companies operating in different sectors, however, is lower but still different from zero because of the influence of a common macroeconomic environment. The second property is crucial because it allows for mutually dependent extremely negative asset returns. And since the structural approach for credit risk modelling explains the failure of a company as its asset value dropping below the value of its outstanding debt, the lower-tail dependence of

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asset returns makes the clustering of default events possible. In turn, joint default events are the main source of tail risk in a portfolio, as measured by Value at Risk or Expected Shortfall.

Summarising, the paper contributes to the existing literature on the (multivariate) stochastic credit risk modelling in the following way:

- It introduces an iterative stochastic time change resulting in a Lévy process with different dependence properties of its multivariate parts. This processes might prove advantageous for the dynamic modelling of asset returns, especially if individual returns are skewed and/or leptokurtic and returns from certain predefined sectors exhibit (an asymmetric) tail dependence.

- The paper investigates the dependence properties of the underlying static marginal copulas. At the lower level of hierarchy, the marginal copulas firstly join single asset returns within given sectors or sub-portfolios, allowing for tail dependence. At the higher level of hierarchy, asset returns from different sectors/sub-portfolios are linked together such that inter-sector dependence is weaker than intra-sector dependence.

- The paper also provides a practical link between the concept of nested Archimedean copulas for joint characteristic functions and the multivariate distributions arising from an iterative stochastic time change, a link which merits further investigation.

- Considering an application to portfolio credit risk modelling in a static setup, the paper illustrates the extent of model risk for the portfolio tail risk measures primarily compared to a Gaussian framework with the same asset correlation.

- For the purposes of credit portfolio modelling, the paper provides an Importance Sampling algorithm for the proposed framework, which may considerably improve simulation efficiency.

Regarding the related literature, I refer to the seminal paper by Madan et al. (1998), in which the authors introduce the Variance Gamma process into the option price literature. I also refer to the papers on using Gamma-time-changed Brownian motion in the *portfolio* context by Luciano and Schoutens (2006) and Moosbrucker (2006), among others. It is worth stressing, however, that those papers which focus on the portfolio settings impose strict restrictions on the choice of dependence parameters. That is, even if certain sub-portfolios can be defined within the portfolio under consideration, the parameters associated with the underlying stochastic time have to be identical across the sub-portfolios. In this paper, however, I use a specific model construction to avoid any such parameter restrictions, allowing the sub-portfolio-specific parameters to be set individually.

In general, Lévy processes provide a convenient framework to model the empirical phenomena from finance: since the sample paths can have jumps, the generating distributions can be fat-tailed and skewed. I refer to Schoutens (2005) and Cont and Tankov (2003) for more useful information on the application of Lévy processes in finance. The particular interest in time-changed Lévy processes for multivariate modelling arises from the hypothesis of common jump arrivals across different assets, which can induce a strong dependence in the tail. This hypothesis was investigated by Bollerslev et al. (2011), who found strong evidence for asymptotic tail dependence in stock returns, with most of it directly attributable to the systematic jump tails and strong dependencies between the sizes of the simultaneously occurring jumps. Although my model does not account for dependencies between the jumps' sizes, this feature can be incorporated by using (positively) correlated Brownian motions at the first modelling stage. I leave this model extension, however, to future research.

The remainder of the paper is structured as follows. Section 2 provides details on the derivation of the dynamic hierarchical model for tail-dependent asset returns, its static copula and on a Monte Carlo sampling algorithm. Section 3 gives an application example and illustrates the model risk in terms of the portfolio tail losses for dependence structures with and without tail dependence. In order to increase efficiency of portfolio tail risk simulation, section 4 elaborates an Importance Sampling algorithm for the VCG setting. Section 5 concludes and summarises the main results of the paper.

2. The Variance Compound Gamma model

I begin this section by deriving a novel multivariate model for logarithmic asset returns which I term Variance Compound Gamma (VCG). The underlying stochastic process is a four-parameter Lévy process designed as a time-changed multivariate uncorrelated Brownian motion with drift. The random time at which the Brownian motion is evaluated is given by an increasing Lévy process (subordinator) termed Compound Gamma (CG). The CG process itself is a time-changed Gamma process evaluated at a random time given by another Gamma process. This specific two-stage time change procedure has a major advantage in that it creates hierarchical dependence between asset return processes.

This section is structured as follows. I introduce the VCG process for dependent asset returns in subsection 2.1. I consider its implications for the static dependence structure in subsection 2.2 and provide a sampling algorithm for the VCG random variables in subsection 2.3. In subsection 2.4 I further investigate the properties of the implicit marginal copulas.

2.1. Multivariate process for logarithmic asset returns

The departing point in the construction of the VCG model are Brownian logarithmic asset returns $\{R_i\}_{t\geq 0}$ for n companies under consideration:

$$R_i(t) = \mu_i \, t + \sigma_i \, W_i(t), \quad i = 1, \dots, n. \tag{2.1}$$

In this equation $\mu \neq 0$ and $\sigma > 0$ are the drift and volatility parameters of the Brownian motion respectively, and $\{W(t)\}_{t\geq 0}$ denotes a Wiener process.

Using Brownian definition (2.1) would have several disadvantages, which have been pointed out by Luciano and Schoutens (2006), among others: a symmetric and mesokurtic distribution of the asset return $R_i(t)$ for all $t > 0$; the almost surely continuous sample paths of the return process $\{R_i(t)\}$ and, hence, anticipated default times; a dependence structure which only allows for linear correlation. To overcome those drawbacks, the authors suggest applying a stochastic time change, i.e. evaluating the Brownian process at a random time. The random time is a stochastic process introduced instead of the deterministic variable t. It can be interpreted as business time, i.e, an information arrival process. When the random time used is a subordinator, its main distinguishing properties are non-decreasing sample paths and stationary and independent increments (see Sato, 1999, p. 137). Therefore, as Luciano and Schoutens put it, three following properties of information flow arise:

- the amount of available information cannot decrease;
- the amount of information released within one period of time only depends on the length of that period;
- the amount of new information is not affected by the information already released.

The authors work out the details in the case of a Gamma-time change, which results in a (pure jump) Variance Gamma process of asset returns. They evaluate all n individual Brownian motions on a common Gamma process introducing a stochastic business time in which the general market operates. That way, dependence properties are identical for all firms in focus, irrespective the different sectors in which they may operate.

I extend the approach described by constructing stochastic business times which are specific to certain groups of firms and are subordinated with respect to the common business time of the general macroeconomic environment. Let $\{Y_j(t)\}_{t\geq 0}$, $j = 1, \ldots, m$, where m denotes the number of sectors in the market, be a set of independent subordinators. A process $\{Y_j(t)\}$ represents the information flow only relevant for the firms operating in the sector j. I time-change each of those processes by evaluating them on a common stochastic time denoted by $\left\{Z^{mrkt}(t)\right\}_{t\geq 0}$. This common subordinator $\left\{Z^{mrkt}(t)\right\}$ represents the general information flow relevant for every firm in the market, irrespective of which industry sector it belongs to. It may be thought of as information about changes in the overall macroeconomic conditions. According to Sato (1999, p. 201), the resulting interdependent Lévy processes $\{Z_j(t)\}_{t\geq 0}$ with

$$Z_j(t) := Y_j\big(Z^{mrkt}(t)\big), \quad j = 1, \ldots, m, \tag{2.2}$$

are again subordinators and thus can act as business times. A business time $\{Z_j(t)\}$ incorporates both the information specific for the sector j and the general macroeconomic information.

Now, let $\{R_{ji}\}_{t\geq 0}$ denote the asset return process of the ith firm in sector j. Than, the model for asset returns can be written as follows:

$$\begin{gathered} R_{ji}(t) = \mu_{ji}\, Z_j(t) + \sigma_{ji}\, W_{ji}\big(Z_j(t)\big), \qquad (2.3) \\ Z_j(t) = Y_j\big(Z^{mrkt}(t)\big), \\ i = 1, \ldots, n_j, \quad j = 1, \ldots, m, \quad \sum\nolimits_{j=1}^{m} n_j = n, \end{gathered}$$

the processes $\{W_{ji}(t)\}$, $\{Y_j(t)\}$ and $\{Z^{mrkt}(t)\}$ being mutually independent. The representation (2.3) can be generalised by adding a drift term $\alpha_{ji}\, t$ with $\alpha_{ji} \neq 0$.

Thus far, the model specification has been kept very general. It holds for all subordinator settings $\{Z^{mrkt}(t)\}$ and $\{Y_j(t)\}$. In the following, though, I provide details on a specific hierarchical model which results from using Gamma subordinators for the business times $\{Z^{mrkt}(t)\}$ and $\{Y_j(t)\}$. The business time $\{Z_j(t)\}$ then arises as a Gamma subordinator evaluated at a Gamma random time and is therefore termed Compound Gamma (CG).

In order to complete the model specification, I need to define parameters of the Gamma processes involved. A Gamma process has independently Gamma-distributed increments characterised by two parameters: the shape parameter $\beta > 0$ and the rate or inverse scale parameter $\lambda > 0$. Thus, for each $t > 0$,

$$Z^{mkrt}(t) \sim \Gamma\big(t\, \beta_{Z^{mkrt}}, \lambda_{Z^{mkrt}}\big), \qquad (2.4)$$

$$Y_j(t) \sim \Gamma\big(t\, \beta_{Y_j}, \lambda_{Y_j}\big). \qquad (2.5)$$

Taking into account the scaling property of a Gamma process $\{X(t)\}$:

$$b\, X(t; \beta, \lambda) \overset{d}{=} X(t; \beta, \lambda/b), \quad b > 0 \quad \forall\, t > 0,$$

a scaling constant b may always be chosen such that $\lambda_{Z^{mkrt}}/b = \beta_{Z^{mkrt}}$ holds in (2.4). Furthermore, because any scaling constant of the subordinator $\{Z^{mkrt}(t)\}$ can be absorbed by the shape parameter of the subordinand $\{Y_j(t)\}$ (to see it, put $bZ^{mkrt}(t)$ instead of t into (2.5)), I define $\beta_{Z^{mkrt}} = \lambda_{Z^{mkrt}}$ without loss of generality. Because of

$$\beta_{Z^{mkrt}} = \frac{1}{t} \frac{E\big[Z^{mrkt}(t)\big]^2}{\operatorname{var}\big(Z^{mrkt}(t)\big)} \quad \text{and} \quad \lambda_{Z^{mkrt}} = \frac{E\big[Z^{mrkt}(t)\big]}{\operatorname{var}\big(Z^{mrkt}(t)\big)},$$

$\beta_{Z^{mkrt}} = \lambda_{Z^{mkrt}}$ implies $E\big[Z^{mrkt}(t)\big] = t$. Thus, the only free parameter we can decide on for the process parametrisation is the variance of the Gamma process $\{Z^{mkrt}(t)\}$ at $t = 1$, which I denote by $\kappa_{Z^{mkrt}}$:

$$Z^{mrkt}(t) \sim \Gamma(t/\kappa_{Z^{mrkt}}, 1/\kappa_{Z^{mrkt}}) \quad \forall t > 0. \qquad (2.6)$$

Based on a similar scaling property of the CG process, and due to the fact that each scaling constant of the CG subordinator can be absorbed by the parameters of the Brownian motion because of

$$b\,W(t) \stackrel{d}{=} W(b^2\,t), \quad b>0 \quad \forall\, t>0,$$

I choose $\beta_{Y_j} = \lambda_{Y_j} = 1/\kappa_{Y_j}$ with $\kappa_{Y_j} := \mathrm{var}\big(Y_j(1)\big)$, i.e.

$$Y_j(t) \sim \Gamma(t/\kappa_{Y_j}, 1/\kappa_{Y_j}) \quad \forall t>0. \tag{2.7}$$

This way, the CG process $\{Z_j(t)\}$ will be characterised by two distribution parameters: $\kappa_{Z^{mrkt}}$ and κ_{Y_j}. One implication of (2.7) is $E\big[Z_j(t)\big] = t$, $j = 1, \ldots, m$, i.e. the stochastic business time equals the physical time in expectation.

For a fixed t the random variable $Z_j(t)$ follows the CG distribution. The CG distribution, denoted here by $f(\cdot)$, can be specified as a mixture of a Gamma density function, denoted by $g(\cdot)$, with a stochastic, Gamma-distributed shape parameter:

$$f\big(x; \kappa_{Z^{mrkt}}, \kappa_{Y_j}\big) = \int_0^\infty g\left(x; \frac{\tau}{\kappa_{Y_j}}, \frac{1}{\kappa_{Y_j}}\right) g\left(\tau; \frac{t}{\kappa_{Z^{mrkt}}}, \frac{1}{\kappa_{Z^{mrkt}}}\right) d\tau. \tag{2.8}$$

Thus, in this paper I use the term "Compound Gamma" in the sense of Giese (2004) and not in the sense of a mixture over the stochastic *scale* parameter as introduced by Dubey (1970).

Even though the CG distribution does not posses a closed-form expression, it is sufficient to know the Laplace transform (LT) of the CG variable $Z_j(t)$. This LT can be derived by means of the identity for the subordinated Lévy processes given in Sato (1999, p. 201):[2]

$$\varphi_{Z_j(t)}(\nu) = \varphi_{Z^{mrkt}(t)}\left[-t^{-1}\ln\big\{\varphi_{Y_j(t)}(\nu)\big\}\right]. \tag{2.9}$$

Since the LT of the Gamma variable $Z^{mrkt}(t)$ is defined as

$$\varphi_{Z^{mrkt}(t)}(\nu) = \big(1 + \nu\kappa_{Z^{mrkt}}\big)^{-t/\kappa_{Z^{mrkt}}}, \tag{2.10}$$

I can write for (2.9):

$$\varphi_{Z_j(t)}(\nu) = \left[1 + \frac{\kappa_{Z^{mrkt}}}{\kappa_{Y_j}}\ln\big(1 + \nu\,\kappa_{Y_j}\big)\right]^{-t/\kappa_{Z^{mrkt}}}. \tag{2.11}$$

Based on (2.11), the characteristic function (cf) of the asset return process defined in (2.3) can be derived according to the formula in Sato (1999, S. 197 f.) for the general case of a Lévy subordination. Because $\{R_{ji}(t)\}_{t\geq 0}$ arises from the Brownian subordination with a CG process, I term this process Variance Compound Gamma (VCG), similarly to the Variance Gamma process introduced into the option pricing literature by Madan et al. (1998). The

[2] $\varphi_X(\cdot)$ denotes the LT of the positive random variable X with $\varphi_X(\nu) = E\big[\exp\{\nu X\}\big]$.

VCG process is a pure-jump Lévy process whose cf is given by:[3]

$$\phi_{R_{ji}(t)}(\theta) = \varphi_{Z_j(t)}\left[-\psi_{X_{ji}}(\theta)\right] \tag{2.12}$$

$$= \left[1 + \frac{\kappa_{Z^{mrkt}}}{\kappa_{Y_j}} \ln\left\{1 - \kappa_{Y_j}\left(i\theta\mu_{ji} - \frac{1}{2}\theta^2\sigma_{ji}^2\right)\right\}\right]^{-t/\kappa_{Z^{mrkt}}}, \tag{2.13}$$

where $i\mu_{ji}\theta - \frac{1}{2}\theta^2\sigma_{ji}^2 =: \psi_{X_{ji}}(\theta)$ is the characteristic exponent of the normal distribution.

The increments of the asset return process $\{R_{ji}(t)\}$ are independently VCG distributed. The VCG distribution arises from a normal mean-variance mixture with a CG mixing probability density (see also equation (2.15) below) and is not known in closed form. Therefore, I only derive an integral expression for the VCG probability distribution function (pdf) in appendix A. Due to its four parameters $\mu_{ji} \in \mathbb{R}, \sigma_{ji}, \kappa_{Y_j}, \kappa_{Z^{mrkt}} \in \mathbb{R}_+$, the VCG distribution possesses a flexible functional form, which is illustrated in Figure 1 for $t = 1$. The plots were obtained by means of inverting the cf (2.13) using a fast Fourier transform algorithm.

The moments of a VCG process can be calculated as polynomials in cumulants. Because the cf is given by a simple closed formula (2.13), cumulants of a VCG process can be obtained as derivatives of the cumulant-generating function defined as the logarithm of the cf. But for the cumulants of higher orders it is easier to use the mixture representation of the random variable $R_{ji}(t)$ and to apply the law of total cumulance introduced by Brillinger (1969). I give the first four moments – mean, variance, skewness and excess kurtosis – in appendix A. Referring to those moments, I describe μ_{ji} as a skewness parameter, σ_{ji} as a variance parameter and κ_{Y_j} and $\kappa_{Z^{mrkt}}$ as kurtosis parameters.

Figure 2 shows a simulated sample path of a Gamma process (a), a realisation of the CG process based thereupon (b) and a realisation of the corresponding VCG process (c). Additionally, I plot in Figure 3 some simulated sample paths of two correlated VCG processes. Since the processes arise as the uncorrelated Brownian motions evaluated on the same CG random time, the jumps occur at identical times, but the direction and the size of the jumps are conditionally independent.

In the next subsection I take a closer look at the static multivariate dependence structure implied by the VCG model.

2.2. Dependence structure

In this subsection, I investigate in detail the dependence structure resulting from the two-stage time change described previously. For this purpose, I focus on the static case of the

[3] $\phi_X(\cdot)$ and $\psi_X(\cdot)$ denote the cf and characteristic exponent of the random variable X with $\phi_X(\theta) = E\left[\exp\{i\theta X\}\right] = \exp\left\{\psi_X(\theta)\right\}$.

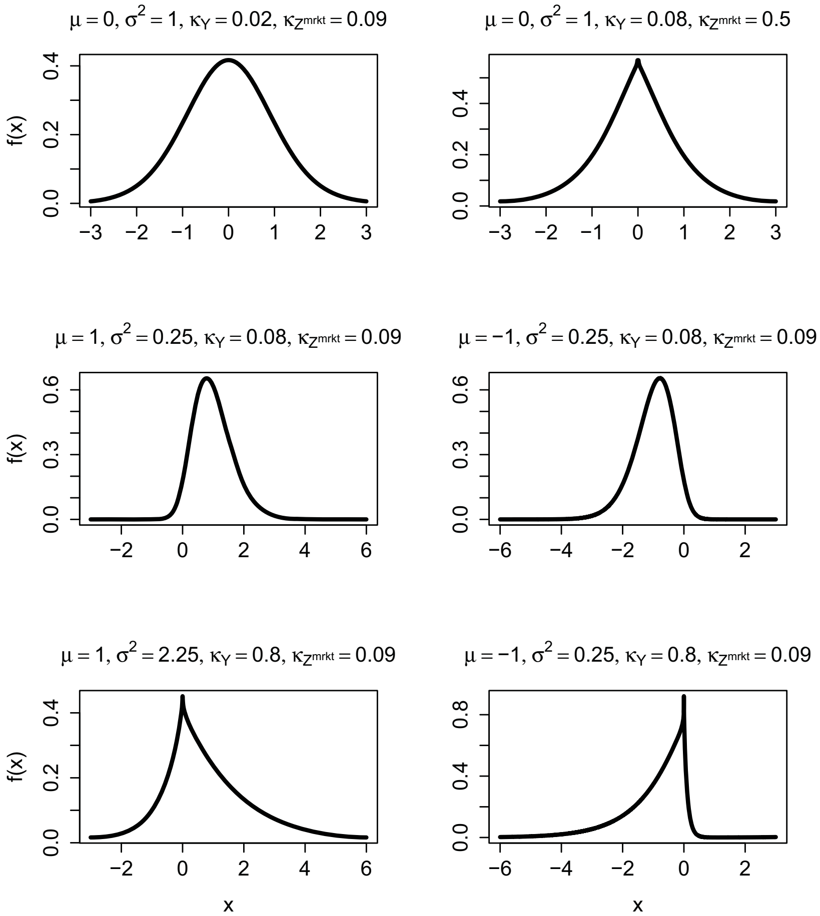


Figure 1: The shape of the Variance Compound Gamma probability distribution function for selected parameters.

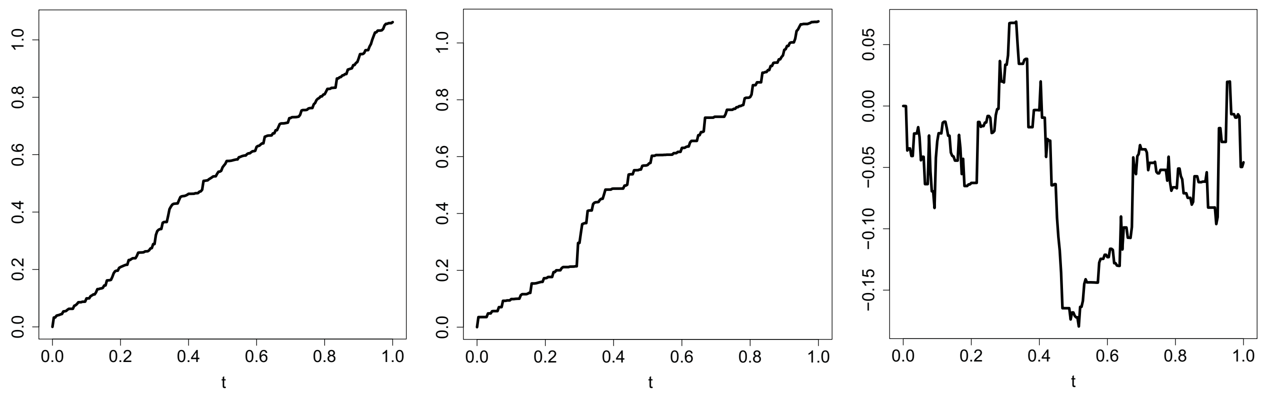
model (2.3) with $t = 1$ and, thus, drop the time index t:

$$R_{ji} = \mu_{ji}\, Z_j + \sigma_{ji}\sqrt{Z_j} W_{ji}. \tag{2.14}$$

This stochastic model has the following mixture representation:

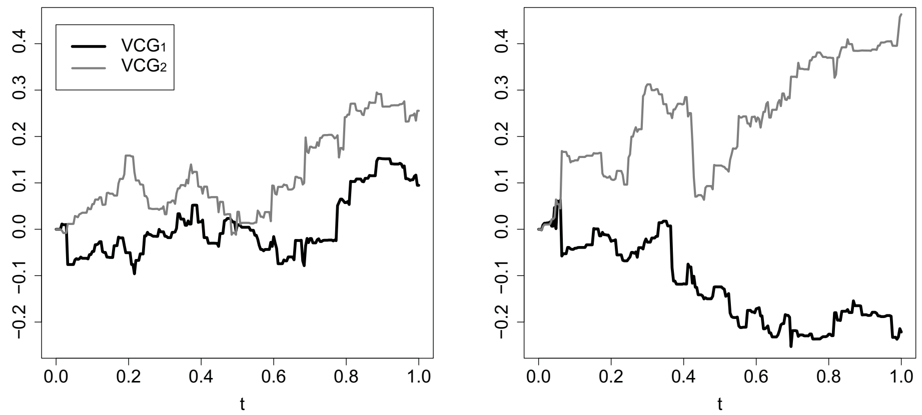
$$\begin{aligned} F_{R_{ji}}(x) &= \int_0^\infty \Phi\left(\frac{x - \mu_{ji}\, z_j}{\sigma_{ji}\sqrt{z_j}}\right)\, dH_{Z_j}(z_j) \\ &\equiv \int_0^\infty \int_0^\infty \Phi\left(\frac{x - \mu_{ji}\, z_j}{\sigma_{ji}\sqrt{z_j}}\right)\, dM_{Z_j|Z^{mrkt}}(z_j \mid z^{mrkt})\, dM_{Z^{mrkt}}(z^{mrkt}). \end{aligned} \tag{2.15}$$

This is the mixture representation of the univariate VCG cumulative distribution function (cdf) $F(\cdot)$ of an asset-return variable. Here I denote a Gamma cdf by $M(\cdot)$, a CG cdf by $H(\cdot)$ and the standard Gaussian cdf by $\Phi(\cdot)$. Notation in form $Y \mid X$ refers to the distribution of a random variable Y conditional on X.



(a) Gamma process (b) CG process (c) VCG process

Figure 2: Simulated sample paths of Gamma, Compound Gamma (CG) and Variance Compound Gamma (VCG) processes with parameters $\mu = -0.02$, $\sigma = 0.2$, $\kappa_Y = 0.01$ and $\kappa_{Z^{mrkt}} = 0.01$. The sample path of the Gamma process was used as a realisation of the stochastic time for the simulation of the CG process. Subsequently, the sample path of the CG process was used for the simulation of the VCG process.



(a) $\mu_1 = -0.02$, $\mu_2 = -0.03$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$ (b) $\mu_1 = -0.02$, $\mu_2 = 0.03$, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$

Figure 3: Simulated sample paths of two correlated Variance Compound Gamma (VCG) processes. Each pair of processes was evaluated based on a realisation of the Compound Gamma stochastic time with parameters $\kappa_Y = 0.01$ and $\kappa_{Z^{mrkt}} = 0.01$.

The hierarchical framework introduced in subsection 2.1 implies that, firstly, conditional on a realisation of the market business time, asset returns of the companies in one sector are independent from those in other sectors. Additionally, when realisations of the sector-specific business times are also known, the asset returns of all companies become mutually independent. Keeping this in mind and taking into account (2.15), I can write the mixture representation of the multivariate asset return distribution as follows:

$$F_{\mathbf{R}}(\mathbf{x}) = \int_0^\infty \int_0^\infty \prod_{i=1}^{n_1} \Phi\left(\frac{x_{1i} - \mu_{1i} z_1}{\sigma_{1i}\sqrt{z_1}}\right) dM_{Z_1|Z^{mrkt}}(z_1 \mid z^{mrkt}) \times \dots \tag{2.16}$$

$$\times \int_0^\infty \prod_{i=1}^{n_m} \Phi\left(\frac{x_{mi} - \mu_{mi} z_m}{\sigma_{mi}\sqrt{z_m}}\right) dM_{Z_m|Z^{mrkt}}(z_m \mid z^{mrkt})\, dM_{Z^{mrkt}}(z^{mrkt}).$$

Although the corresponding pdf can only be given in its integral representation (see appendix A), the joint cf of asset returns has a relatively simple form (cf. (2.9) to (2.13)):

$$\phi_{R_{11},\dots,R_{mn_m}}(\theta_{11},\dots,\theta_{mn_m}) = E_{Z^{mrkt}}\left[\prod_{j=1}^{m} E_{Z_j|Z^{mrkt}}\left[\prod_{i=1}^{n_j} \phi_{R_{ji}|Z_j}(\theta_{ji})\right]\right] \tag{2.17}$$

$$= \varphi_{Z^{mrkt}}\left[-\sum_{j=1}^{m} \ln\left\{\varphi_{Y_j}\left(-\sum_{i=1}^{n_j} \psi_{X_{ji}}(\theta_{ji})\right)\right\}\right]$$

$$= \left[1 + \kappa_{Z^{mrkt}} \sum_{j=1}^{m} \frac{1}{\kappa_{Y_j}} \ln\left\{1 - \kappa_{Y_j} \sum_{i=1}^{n_j}\left(i\theta_{ji}\mu_{ji} - \tfrac{1}{2}\theta_{ji}^2\sigma_{ji}^2\right)\right\}\right]^{-1/\kappa_{Z^{mrkt}}} \tag{2.18}$$

The linear dependence between VCG-distributed asset returns can be described by means of the Pearson's correlation coefficient. Correlation between the asset returns of any two firms arises from the common business time they are evaluated at. Within a sector j both the general market and the sector-specific stochastic times take part in the covariation of asset returns. The corresponding correlation coefficient is given by:

$$\operatorname{corr}(R_{ji}, R_{jk}) = \frac{\mu_{ji}\,\mu_{jk}\,(\kappa_{Z^{mrkt}} + \kappa_{Y_j})}{\sqrt{\sigma_{ji}^2 + \mu_{ji}^2(\kappa_{Z^{mrkt}} + \kappa_{Y_j})}\sqrt{\sigma_{jk}^2 + \mu_{jk}^2(\kappa_{Z^{mrkt}} + \kappa_{Y_j})}}. \tag{2.19}$$

For any two firms which belong to distinct sectors j and l, the covariation is only due to the market business time, and the correlation coefficient is given by:

$$\operatorname{corr}(R_{ji}, R_{lk}) = \frac{\mu_{ji}\,\mu_{lk}\,\kappa_{Z^{mrkt}}}{\sqrt{\sigma_{ji}^2 + \mu_{ji}^2(\kappa_{Z^{mrkt}} + \kappa_{Y_j})}\sqrt{\sigma_{lk}^2 + \mu_{lk}^2(\kappa_{Z^{mrkt}} + \kappa_{Y_l})}}. \tag{2.20}$$

When both skewness parameters have the same sign, the asset returns are positively correlated. Otherwise they are negatively correlated. If at least one of the firms in a pair under consideration has a zero skewness parameter, the asset returns are uncorrelated. Nevertheless, they are still associated, since they are driven by a common factor. Apart from the skewness parameters, the magnitude of the linear correlation coefficient also depends on the variance parameters of the Gamma variables: the smaller a parameter $\kappa_{(\cdot)}$, i.e. the closer the corresponding business time is to the physical time, the smaller the correlation between two asset returns.

As for the non-linear dependence structure, the overall, intra-sector and inter-sector copulas implied by the VCG model of asset returns can only be specified implicitly. The overall implicit copula, which joins n marginal VCG distributions to the multivariate distribution given in (2.16), has the general form:

$$C(u_{11}, \ldots, u_{mn_m}) = F_{\mathbf{R}}\big(F^{-1}_{R_{11}}(u_{11}), \ldots, F^{-1}_{R_{mnm}}(u_{mn_m})\big). \tag{2.21}$$

The arguments u_{ji} are probability-integral transforms of R_{ji}: $u_{ji} = F_{R_{ji}}\big(R_{ji} = x_{ji}\big)$. I term the function in (2.21) the hierarchical VCG copula.

Two special cases of marginal copulas of (2.21) are of interest. Firstly, the n_j-dimensional marginal copula of the asset returns $\mathbf{R}_j$, all of which belong to the same sector j:

$$C(u_{j1}, \ldots, u_{jn_j}) = F_{\mathbf{R}_j}\big(F^{-1}_{R_{j1}}(u_{j1}), \ldots, F^{-1}_{R_{jn_j}}(u_{jn_j})\big). \tag{2.22}$$

This copula links marginal VCG distributions which share the same sector-specific business time Z_j, to the multivariate VCG distribution of vector $\mathbf{R}_j$, with a joint cdf given by

$$\begin{aligned} F_{\mathbf{R}_j}(\mathbf{x}_j) &= \int_0^\infty \prod_{i=1}^{n_j} F_{R_{ji}|Z_j}(x_{ji})\, dH_{Z_j}(z_j) \\ &= \int_0^\infty \prod_{i=1}^{n_j} \Phi\left(\frac{x_{ji} - \mu_{ji} z_j}{\sigma_{ji}\sqrt{z_j}}\right)\, dH_{Z_j}(z_j). \end{aligned} \tag{2.23}$$

I term the implicit *intra-sector* copula (2.22) the VCG copula.

The second special case is the marginal copula of asset returns $\mathbf{R}_i$, each of which belongs to a different sector:

$$C(u_{1i}, \ldots, u_{m\,i}) = F_{\mathbf{R}_i}\big(F^{-1}_{R_{1i}}(u_{1i}), \ldots, F^{-1}_{R_{m\,i}}(u_{m\,i})\big). \tag{2.24}$$

This copula joins marginal VCG distributions associated with *different* sector-specific business times Z_j to a multivariate grouped VCG distribution given by[4]

$$F_{\mathbf{R}_i}(\mathbf{x}_i) = \int_0^\infty \prod_{j=1}^{m} F_{R_{ji}|Z^{mrkt}}(x_{ji})\, dM_{Z^{mrkt}}(z^{mrkt}) \tag{2.25}$$

$$= \int_0^\infty \prod_{j=1}^{m} \int_0^\infty \Phi\left(\frac{x_{ji} - \mu_{ji} z_j}{\sigma_{ji}\sqrt{z_j}}\right)\, g\left(z_j; \frac{z^{mrkt}}{\kappa_{Y_j}}, \frac{1}{\kappa_{Y_j}}\right) dz_j \cdot g\left(z^{mrkt}; \frac{1}{\kappa_{Z^{mrkt}}}, \frac{1}{\kappa_{Z^{mrkt}}}\right) dz^{mrkt}.$$

I term the implicit *inter-sector* copula (2.24) the grouped VCG copula.

[4]Note that I do not use the term "grouped" copula in the sense of Daul et al. (2003), who introduced a meta t-distribution based on *comonotone* common factors.

Whereas there is no closed solution for the copulas implied by the multivariate VCG setting, it is easy to define explicit copulas for the corresponding joint cfs. In fact, they can be specified in terms of the Hierarchical Archimedean Copulas (HAC), as shown in appendix B. Interestingly, already Wang (1999) pointed out that copula formulas cab be applied to marginal cfs in order to obtain some new multivariate distributions. This would also allow application of efficient numerical fast Fourier transform techniques for calculating the aggregate loss distribution of correlated risks.

I adapt the copula concept to the marginal cfs of asset returns. Defining the arguments of the copulas as $\nu_{ji} := \phi_{R_{ji}}(\theta_{ji})$ (see (2.13)) instead of u_{ji}, I obtain the following HAC representation of the overall cf (2.18):

$$C^{Z^{mrkt}}\left(C^{Z_1}\left(v_{11},\ldots,v_{1n_1}\right),\ldots,C^{Z_m}\left(v_{m1},\ldots,v_{mn_m}\right)\right)$$
$$=\varphi_{Z^{mrkt}}\left[\sum_{j=1}^{m}\varphi^{-1}_{Z^{mrkt}}\circ\varphi_{Z_j}\left(\sum_{i=1}^{n_j}\varphi^{-1}_{Z_j}(v_{ji})\right)\right]. \tag{2.26}$$

The corresponding n_j-dimensional marginal Archimedean copula for the companies belonging to the same sector j is then given by:

$$C^{Z_j}\left(v_{j1},\ldots,v_{jn_j}\right)=\varphi_{Z_j}\left[\sum_{i=1}^{n_j}\varphi^{-1}_{Z_j}(v_{ji})\right] \tag{2.27}$$

and the copula for the companies belonging to different sectors is given by:

$$C^{Z^{mrkt}}\left(v_{1i},\ldots,v_{m\,i}\right)=\varphi_{Z^{mrkt}}\left[\sum_{j=1}^{m}\varphi^{-1}_{Z^{mrkt}}(v_{ji})\right]. \tag{2.28}$$

The functions (2.26) to (2.28) turn out to be exactly the same as the conventional (nested) Archimedean copulas which arise from a Gamma-mixture of powers introduced by Puzanova (2011).

Before I proceed to the properties of the copulas introduced, I first provide a sampling algorithm for the hierarchial VCG model, which I will use for simulation purposes.

2.3. Sampling

In order to generate realisations from the hierarchical VCG model with the joint distribution function defined in (2.16), only algorithms for the simulation of Gamma and normal random variables are needed. Such algorithms belong to the standard configuration of statistical and mathematical software.

Sampling algorithm 1: Monte Carlo for VCG asset returns

- Sample $Z^{mrkt} \sim \Gamma(1/\kappa_{Z^{mrkt}}, 1/\kappa_{Z^{mrkt}})$.
- Sample $Z_j \mid Z^{mrkt}$, $j = 1, \ldots, m$ from the independent Gamma distributions with parameters $\left(Z^{mrkt}/\kappa_{Y_j}, 1/\kappa_{Y_j}\right)$.
- Sample $W_{ji} \overset{iid}{\sim} N(0,1)$, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$.
- The VCG realisations are given by
  $R_{ji} = \mu_{ji} \cdot \left(Z_j \mid Z^{mrkt}\right) + \sigma_{ji}\sqrt{Z_j \mid Z^{mrkt}}W_{ji}$.

2.4. Copula properties

To investigate the dependence properties of the copulas implied by the VCG model and to attain some feeling for their sensitivity with respect to the parameters, I first use the information provided by contour and scatter plots for the two-dimensional marginal copulas.

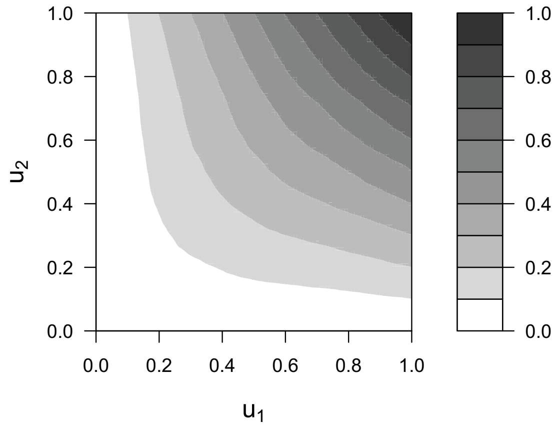
Since a copula function is invariant under any strictly increasing transformation of the marginal distributions, I begin with the standardisation of asset returns. Using the expectation and variance given in (A.1) and (A.2) respectively, I can write the following stochastic representation for standardised asset returns $\tilde{R}_{ji}$:

$$\tilde{R}_{ji} = -\mu_{ji} + \mu_{ji}Z_j + \sqrt{Z_j\left[1 - \mu_{ji}^2(\kappa_{Z^{mrkt}} + \kappa_{Y_j})\right]}W_{ji}. \tag{2.29}$$

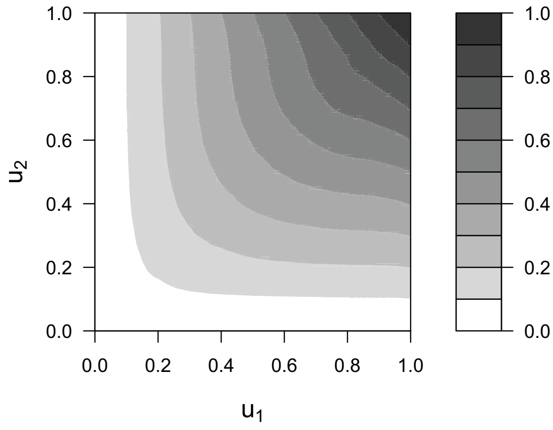
Now the location parameter is different from zero: it equals $-\mu_{ji}$. The scale parameter of the Gaussian part σ_{ji} has no effect on the dependence structure: it disappears in the course of the standardisation procedure.

To produce a contour plot of a two-dimensional VCG copula according to representation (2.22), I make use of the cf of VCG returns, which can be inverted numerically by means of fast Fourier transform. The results for various parameter settings are plotted in Figure 4. For the usual case of negatively skewed asset returns, the plotted level curves lie between those of the maximum copula and independence copula (i.e. the copula of comonotone random variables): cf. Figure 5. Moreover, the positive dependence is stronger for larger values of the shape parameters.

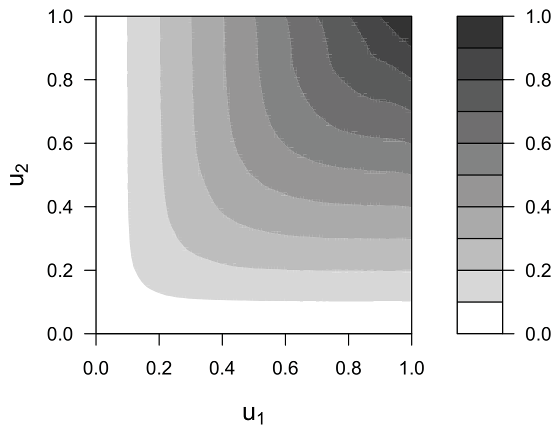
In addition to this information, the scatter plots in Figure 6, generated by simulation, provide insight into how strongly the underlying random variables are associated. On the one hand, we observe more points on the increasing diagonal for the larger parameter values, which is evidence for a stronger positive association. On the other hand, the pairwise realisations cluster in the lower left-hand corner of a unit square, indicating lower-tail dependence which is stronger for larger values of the skewness parameters μ.



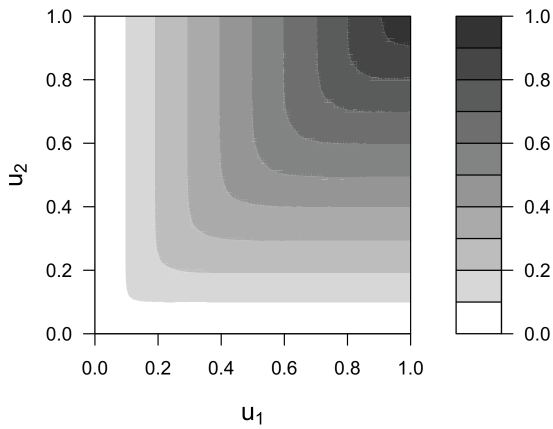
(a) $\mu_1 = \mu_2 = -0.5$, $\kappa_Y = 0.3$, $\kappa_{Z^{mrkt}} = 0.2$



(b) $\mu_1 = \mu_2 = -0.8$, $\kappa_Y = 0.5$, $\kappa_{Z^{mrkt}} = 0.5$



(c) $\mu_1 = \mu_2 = -0.9$, $\kappa_Y = 0.7$, $\kappa_{Z^{mrkt}} = 0.3$



(d) $\mu_1 = \mu_2 = -0.9$, $\kappa_Y = 0.9$, $\kappa_{Z^{mrkt}} = 0.3$

Figure 4: Contour plots of a Variance Compound Gamma copula for different parameter values. Obtained using numerical techniques.

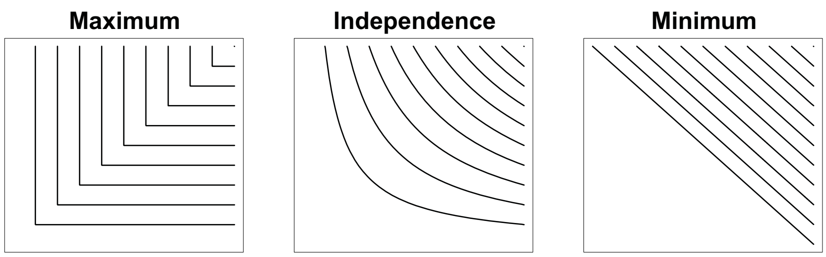
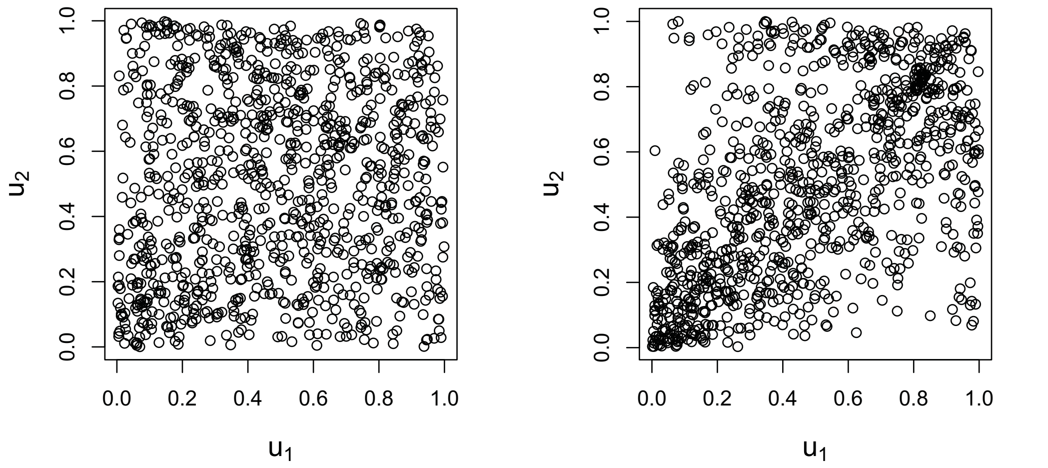
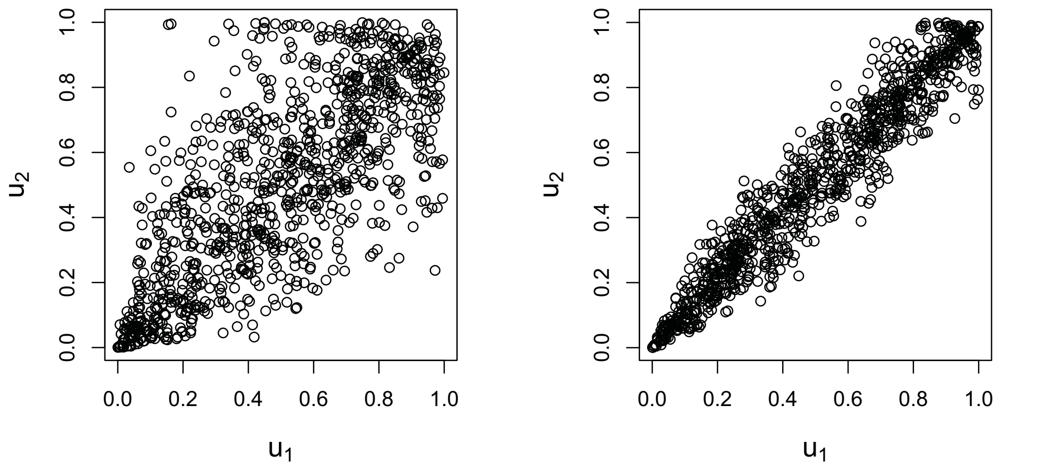


Figure 5: Contour plots of the maximum, independence and minimum copulas.

It is worth noting that the (tail) dependence properties of the jointly VCG distributed variables change depending on the sign of the skewness parameters. If both skewness parameters are positive, the underlying random variables are still positively dependent and the contour plots do not change, but the points in the scatter plots cluster in the upper right-hand corner of a unit square (a 180° rotation), indicating upper tail dependence. If only one of the skewness parameters is positive, the contour lines lie between those of the independence and minimum copula (i.e. the copula of countermonotone random variables; see Figure 5) since the underlying random variables are negatively dependent. As for the scatter plots, they undergo a reflection across either the horizontal (if $\mu_1 < 0$ and $\mu_2 > 0$) or vertical (if $\mu_1 > 0$ and $\mu_2 < 0$) line which passes through the point 0.5.



(a) $\mu_1 = \mu_2 = -0.5$, $\kappa_Y = 0.3$, $\kappa_{Z^{mrkt}} = 0.2$ (b) $\mu_1 = \mu_2 = -0.8$, $\kappa_Y = 0.5$, $\kappa_{Z^{mrkt}} = 0.5$



(c) $\mu_1 = \mu_2 = -0.9$, $\kappa_Y = 0.7$, $\kappa_{Z^{mrkt}} = 0.3$ (d) $\mu_1 = \mu_2 = -0.9$, $\kappa_Y = 0.9$, $\kappa_{Z^{mrkt}} = 0.3$

Figure 6: Scatter plots of 1,000 realisations from the Variance Compound Gamma copula for different parameter values.

The lower-tail dependence property of the VCG asset return model can be interpreted as the tendency of extreme, negative asset returns to occur simultaneously, e.g. during market crashes and economic downturns. In statistical terms, the coefficient of lower-tail dependence, λ_L (upper-tail dependence, λ_U) expresses the limiting conditional probability of the joint exceedance of a lower (upper) quantile. For continues cdfs, those coefficients are given as

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u,u)}{u}, \tag{2.30}$$

$$\lambda_U = \lim_{u \uparrow 1} \frac{1 - 2u + C(u,u)}{1-u}, \tag{2.31}$$

provided the limits exist. If the corresponding limit lies in the interval $(0, 1]$ than the copula exhibits tail dependence.

Since the tail dependence coefficients for the VCG copula cannot be computed analytically, I use the sample versions of λ_L and λ_U in order to gauge the magnitude of the tail dependence. The sample estimates based on formulae (2.30) and (2.31) are discussed, for example, in Schmidt and Stadtmüller (2006). Let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}$ denote a random sample of s observations from a vector (X, Y). Let $k \in (0, 1)$ be the threshold parameter to be chosen by the statistician. Then the sample version of the tail dependence parameters can be represented as:[5]

$$\widehat{\lambda}_L = \frac{1}{k \cdot s} \sum_{i=1}^{s} \mathbb{1}\big(Rank(x_i) \leq k \cdot s \text{ and } Rank(y_i) \leq k \cdot s\big), \tag{2.32}$$

$$\widehat{\lambda}_U = \frac{1}{k \cdot s} \sum_{i=1}^{s} \mathbb{1}\big(Rank(x_i) > s - k \cdot s \text{ and } Rank(y_i) > s - k \cdot s\big).$$

In Figure 7 I illustrate the sample tail dependence coefficients for two parameter settings. In both cases the estimate of λ_L shown for $u < 0.5$ converges to a value greater than zero, indicating positive lower-tail dependence. For $u > 0.5$ the estimate of λ_U converges to zero, indicating no upper-tail dependence.

In addition to the graphical illustrations, I report in Table 1 the sample version of the concordance measure known as Kendall's tau and the estimates of the lower-tail dependence coefficient for the two-dimensional VCG copula for a wider range of parameters. Both estimates are calculated on the basis of a simulated sample of jointly VCG distributed returns.

Let us first consider the concordance. Again, let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}$ denote a random sample of s observations from a vector (X, Y). Then pairs (x_i, y_i) and (x_j, y_j) are concordant if either $x_i < x_j$ and $y_i < y_j$, or $x_i > x_j$ and $y_i > y_j$. They are discordant if either $x_i < x_j$ and $y_i > y_j$, or $x_i > x_j$ and $y_i < y_j$. Nelsen (1999, pp. 125-126) explains that a pair of random variables is concordant if "large" values of one tend to be associated with "large" values of the other and gives the sample version of Kendall's tau as

$$\widehat{\tau} = \frac{c - d}{c + d},$$

[5] $\mathbb{1}(A)$ is an indicator function which equals one if the condition A is true and zero otherwise.

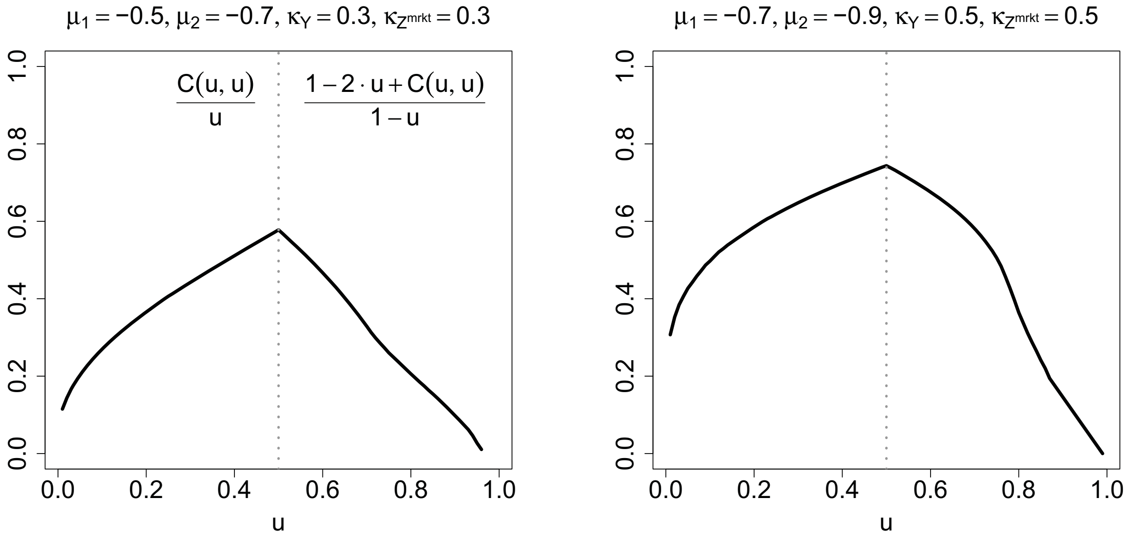


Figure 7: A graphical representation of the tail dependence parameters for a two-dimensional Variance Compound Gamma (VCG) copula for two different parameter sets. Lower and upper tail dependence coefficients are given for $u < 0.5$ and $u > 0.5$ respectively, based on 10^6 simulated realisations from the VCG copula.

where c denotes the number of concordant pairs in the random sample, and d denotes the number of discordant pairs. The consistent monotonically increasing pattern in the estimates of τ listed in the left-hand panel of Table 1 indicates that the VCG copula family is ordered with respect to each of the parameters μ, κ_Y and $\kappa_{Z^{mrkt}}$ in the sense of concordance ordering.

Secondly, we will turn to the lower-tail dependence property. I use expression (2.32) for the estimation of the lower-tail dependence coefficient and set the threshold value k to 1%. That is, for the estimation I only use 1% of the smallest realisations. The right-hand panel of Table 1 shows that the magnitude of lower-tail dependence for the two-dimensional VCG copula is increasing in the copula parameters.

Overall, let me conclude that the degree of positive dependence and the dependence of extremely negative realisations of jointly VCG distributed random variables both increase along with increasing values of skewness and kurtosis parameters. An evident increase in positive dependence which goes along with rising variance of the common CG mixing variable given by $\mathrm{var}(Z_j) = \kappa_{Z^{mrkt}} + \kappa_{Y_j}$ is due to the growing impact of the stochastic business time on the otherwise independent random variables. An increase in the absolute values of the negative skewness parameters goes along with a pronounced rise in lower-tail dependence since negative realisations of both associated random variables are more likely to occur together.

As for the two-dimensional grouped VCG copula, which is the copula of asset returns of companies operating in two different sectors, I abstain from a graphical representation of its dependence properties. The copula's properties do not depend on the sector-specific parameter κ_Y. Moreover, in simulation studies for this copula I could not identify any evidence of lower-tail dependence. Apart from that, the inter-sector, grouped VCG copula is smaller (in

Table 1: Kendall's tau and lower-tail dependence for the VCG copula

		$\widehat{\tau}$			$\widehat{\lambda}_L$		
$\kappa_{Z^{mrkt}}$	μ	$\kappa_Y = 0.2$	$\kappa_Y = 0.5$	$\kappa_Y = 0.9$	$\kappa_Y = 0.2$	$\kappa_Y = 0.5$	$\kappa_Y = 0.9$
0.1	−0.5	0.0450	0.0793	0.1210	0.0624	0.0970	0.1457
	−0.7	0.0841	0.1610	0.2687	0.0766	0.1521	0.2491
	−0.9	0.1420	0.2926	0.5581	0.1076	0.2327	0.4994
0.2	−0.5	0.0543	0.0887	0.1328	0.0706	0.1098	0.1610
	−0.7	0.1151	0.1892	0.3017	0.1010	0.1710	0.2743
	−0.9	0.1935	0.3502	0.6690	0.1465	0.2700	0.6013
0.3	−0.5	0.0671	0.1001	0.1445	0.0797	0.1170	0.1703
	−0.7	0.1408	0.2181	0.3324	0.1191	0.1913	0.3004
	−0.9	0.2420	0.4160	0.8351	0.1809	0.3209	0.7896

Note: The sample versions of Kendall's tau (τ) and of the lower-tail dependence coefficient (λ_L) are given for the two-dimensional Variance Compound Gamma (VCG) copula for various parameter settings. Parameter μ is always identical for a pair of random variables. Each estimate is computed based on 10^6 realisations of the jointly VCG distributed standardised variables specified in (2.29). I use 1% of the smallest realisations in order to estimate λ_L.

the sense of concordance ordering) than the intra-sector VCG copula for two VCG random variables, as can be seen by comparing the results in Table 2 with those in Table 1. Because of the consistent monotonically increasing pattern in the estimates of τ listed in Table 2, I conclude that the grouped VCG copula family is ordered with respect to each of the parameters μ and $\kappa_{Z^{mrkt}}$.

Finally, addressing the issue of model risk, I show by means of a graphical representation in Figure 8 that the tail behaviour of two models with exactly the same correlation of asset returns may be quite different. I compare realisations of two jointly normally distributed random variables with those of the VCG-distributed variables. I calibrate the parameters of the bivariate VCG distribution so as to achieve the level of linear correlation specified for the Gaussian model. In the two upper scatter plots of uncorrelated asset returns we observe that, under the VCG distribution assumptions, large negative and positive realisations are more likely due to positive excess kurtosis driven by variance parameters of Gamma random times. For the positive correlation (the two lower scatter plots), negative skewness of VCG random variables leads to lower-tail dependence not observed in the Gaussian case. Applied for the purposes of credit risk modelling, this implies more joint default events and larger credit portfolio losses, as will be seen in the next section.

Table 2: Kendall's tau for the grouped VCG copula

	$\widehat{\tau}$		
$\kappa_{Z^{mrkt}}$	$\mu = -0.5$	$\mu = -0.7$	$\mu = -0.9$
0.1	0.0132	0.0291	0.0524
0.2	0.0291	0.0556	0.1056
0.3	0.0413	0.0851	0.1606

Note: The sample version of Kendall's tau (τ) is given for the two-dimensional grouped Variance Compound Gamma (VCG) copula. Each estimate is computed based on 10^6 realisations of the jointly grouped VCG distributed standardised variables specified in (2.29). The estimates reported here are those for $\kappa_Y = 0.5$ and various values of parameters $\kappa_{Z^{mrkt}}$ and μ. Parameters κ_Y and μ are always identical for a pair of random variables.

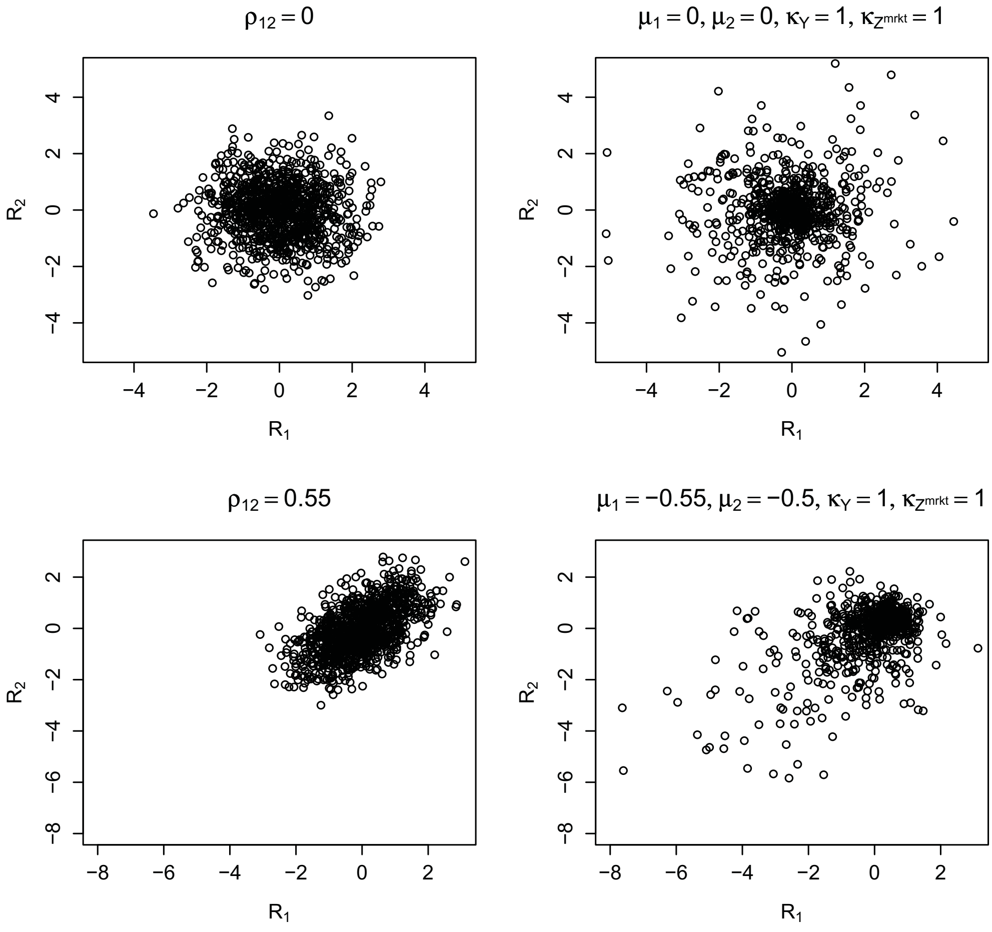


Figure 8: Comparison of scatter plots of 1,000 realisations of standardised, normally (left-hand column) and Variance Compound Gamma, VCG (right-hand column) distributed random variables R_1 and R_2 for different parameter values. The VCG parameters are calibrated so as to ensure the same linear correlation in a respective row.

3. An application example

In this section I carry out a simulation study for two credit portfolios in order to show the extent of model risk in terms of significant differences in the tail risk measures obtained for different models of asset returns with identical linear correlation structure but different tail dependence properties.

The section is organised as follows. In subsection 3.1 the variables of interest are defined: portfolio loss rate, Value at Risk (VaR) and Expected Shortfall (ES). Then the test portfolios and the model parameters used for the simulation exercise are specified in subsections 3.2 and 3.3 respectively. Finally, simulation results for different model settings are discussed in subsection 3.4.

3.1. Portfolio setup

In the tradition of the default-only credit risk models, I look at the probability distribution of the portfolio losses at the one-year risk horizon, whereas the losses can only materialise if one or more borrowers default on their obligations. Let n denote the number of borrowers in the portfolio. For each borrower i there is one, and only one, (aggregate) credit exposure i. The borrower's loss given default in monetary units equals its potential exposure at default less the expected recovery. This loss given default of the borrower i divided by the loss given default of all borrowers in portfolio results in the loss given default rate denoted by LGD_i. The portfolio loss rate denoted PL is a random variable defined as the sum over the individual loss rates L_i, with L_i being equal to zero when the ith borrower survives beyond the risk horizon and equal to LGD_i when the borrower defaults on its obligations.

The event of a borrower i's default is determined in the tradition of structural credit risk models by the standardised returns on the borrower's market value of assets $\tilde{R}_i$ falling below the default threshold. The default threshold is defined by the borrower's one-year probability of default PD_i and equals $F^{-1}_{\tilde{R}_i}(PD_i)$, $F^{-1}_{\tilde{R}_i}$ being the quantile function of $\tilde{R}_i$. The portfolio setting can be summarised as follows[6]:

$$PL := \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} LGD_i \cdot \mathbb{1}\Big(\tilde{R}_i \leq F^{-1}_{\tilde{R}_i}(PD_i)\Big) \tag{3.1}$$

With regard to the distribution of the portfolio loss rate PL, I am looking for the Value at Risk at a pre-specified confidence level q (VaR_q) and for the Expected Shortfall (ES_q). VaR is commonly used in risk management and controlling as a measure of portfolio credit risk, although it is incoherent (not sub-additive in general; see Acerbi and Tasche, 2001). It quantifies the *minimum* portfolio loss in the worst $(1-q) \times 100$ per cent of cases. $VaR_q(PL)$

[6] In this subsection, I drop the sector subscript j for simplicity.

equals the value of the quantile function of the random variable PL:

$$VaR_q(PL) := F_{PL}^{-1}(q) \tag{3.2}$$

ES is a coherent risk measure which quantifies the *expected* portfolio loss in the worst $(1 - q) \times 100$ per cent of cases. $ES_q(PL)$ equals the conditional tail expectation beyond the q-quantile of the portfolio loss distribution augmented by a discontinuity adjustment (Acerbi and Tasche, 2001):

$$ES_q(PL) := E\big[PL \mid PL \geq VaR_q(PL)\big] + VaR_q(PL) \cdot \frac{1 - q - \Pr\{PL \geq VaR_q(PL)\}}{1 - q}. \tag{3.3}$$

Portfolio loss rate distribution can be estimated by means of simulation. First, I sample standardised asset returns $\tilde{R}_{ji}$ according to sampling algorithm 1. On that basis, I calculate default indicators as

$$D_i := \mathbb{1}\Big(\tilde{R}_i \leq F_{\tilde{R}_i}^{-1}(PD_i)\Big). \tag{3.4}$$

For computation of the default barrier $F_{\tilde{R}_i}^{-1}(PD_i)$ within VCG settings, I use a fast Fourier transform algorithm implemented in R and invert the univariate VCG cf numerically.

After s simulation runs I compute Monte Carlo estimators for the portfolio loss distribution, VaR and ES given by:

$$\begin{aligned}
&\hat{F}_{PL}(x_q) \equiv \hat{q} = \frac{1}{s}\sum_{k=1}^{s} \mathbb{1}_{(0,x_q]}(PL^k), \\
&\widehat{VaR}_q(PL) = \inf\big\{x \in [0,1] \;:\; \hat{F}_{PL}(x) \geq q\big\} = PL^s_{\lceil s \cdot q \rceil}, \\
&\widehat{ES}_q(PL) = \frac{\sum_{k=1}^{s} PL^k\, \mathbb{1}_{\left(\widehat{VaR}_q(PL),1\right]}(PL^k)}{\sum_{k=1}^{s} \mathbb{1}_{\left(\widehat{VaR}_q(PL),1\right]}(PL^k)} \\
&+ \widehat{VaR}_q(PL) \frac{1 - \hat{q} - \frac{1}{s}\sum_{k=1}^{s} \mathbb{1}_{\left(\widehat{VaR}_q(PL),1\right]}(PL^k)}{1 - \hat{q}}
\end{aligned}$$

respectively. Here $PL^s_{\lceil s \cdot q \rceil}$ represents the order statistic of the sample $\{PL^1, \ldots, PL^s\}$ which is either of order $s \cdot q$ or a larger order next to it.

3.2. Test portfolios

For the sake of comparability, I use the same two stylised portfolios as in Puzanova (2011) so that I can collate results on the portfolio tail loss based on VCG settings with those for the Hierarchical Archimedean Copula (HAC) model.[7] Each portfolio consists of only two

[7] Currently, the HAC model is only available in a static form, i.e. in contrast to the VCG model there is no HAC representation in terms of stochastic processes governing asset returns. The HAC has the same

Table 3: Structure of the small test portfolio

Rating category	PD(%)	Share in the total portfolio LGD(%)	Total # of debtors
IG: Aa	0.064	35	10
IG: A	0.077	15	10
IG: Baa	0.301	15	25
SG: Ba	1.394	15	25
SG: B	4.477	15	25
SG: C	14.692	5	5

Note: Rating categories and corresponding probabilities of default (PD) were obtained from Moody's (2006, p. 33). IG indicates investment-grade ratings and SG indicates speculative-grade ratings. Composition of the portfolio is the same as in Puzanova et al. (2009).

sub-portfolios corresponding to two sectors $j = 1, 2$.

The smaller portfolio is comprised of 100 credit exposures as summarised in Table 3. The portfolio LGD is attributed to 65% to investment-grade borrowers (IG, sector $j = 1$) and to 35% to speculative-grade borrowers (SG, sector $j = 2$) according to Moody's rating grades/categories. The broad rating grades IG and SG serve in our example as two sectors with different sector-specific dependence parameters. In each rating category 80% of the total LGD is evenly distributed among 20% of the largest debtors. The remaining 20% of the LGD in each rating category is evenly distributed among the remaining debtors. The second, larger portfolio comprised of 1,000 credit exposures has the identical PD-LGD structure to that represented in Table 3 and is obtained from the small portfolio by subdividing each credit exposure into 10 parts.

3.3. Parameter setup

As mentioned before, I will compare two hierarchical modelling frameworks with lower-tail dependence (VCG and HAC) with a Gaussian model. In order to lay down a benchmark Gaussian specification, I modify the one-factor Gaussian model of the Vasicek type (Vasicek, 1987) accordingly. At the top, market (or portfolio) level of the hierarchy, all obligors in portfolio are related to each other through the systematic factor Z^{mrkt}, which specifies the inter-group co-variation. At the lower level of the hierarchy, the sector-specific (or sub-portfolio-specific) systematic factors Y_j specify the additional intra-group co-variation. The remaining variation of asset returns is attributed to an idiosyncratic component W_{ji}. To put

form as on the right-hand side of (2.17), when applied to probability-integral transforms of asset returns. In the present paper, the nested copula (2.17), however, links single cfs to the joint cf, which results in a joint distribution of asset returns whose hierarchical VCG copula can only be given implicitly.

Table 4: Model parameters used in the simulation

Model	Parameters	Estimates
Gauss	ρ_1	0.0321
	ρ_2	0.1212
	ρ_{mrkt}	0.0144
VCG/HAC	κ_{Y_1}	0.0214
	κ_{Y_2}	0.1309
	$\kappa_{Z^{mrkt}}$	0.0175
	μ_1	-0.9084
	μ_2	-0.9036

Note: The correlation parameters used in the simulation for the Gaussian model are set following Puzanova and Siddiqui (2005). The model parameters for the Variance Compound Gamma model (VCG) and Hierarchical Archimedean Copula (HAC) are calibrated accordingly in order to maintain the same linear correlation structure.

it formally, the Gaussian model for standardised asset returns turns out to be:

$$\tilde{R}^G_{ji} = \sqrt{\rho_j - \rho_{mrkt}}Y_j + \sqrt{\rho_{mrkt}}Z^{mrkt} + \sqrt{1-\rho_j}W_{ji}, \quad j = 1, 2. \tag{3.5}$$

According to (3.5) the intra-sector correlation in the Gaussian setting equals ρ_j, whereas the inter-sector correlation is given by ρ_{mrkt}. These linear correlation coefficients are also binding for other two portfolio models under consideration.

For the sake of comparability, I set the estimates of the kurtosis parameters $\kappa_{Z^{mrkt}}$ and κ_{Y_j} as given in Puzanova (2011). Furthermore, I simplify the issue of parameter calibration for the VCG model by using identical skewness parameters for obligors belonging to the same sector j, i.e. $\mu_{ji} \equiv \mu_j$. I set $\mu_j = \sqrt{\rho_j/\left(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\right)}$ (cf. (2.19)) to meet the required intra-group asset correlation. The inter-group asset correlation equals ρ_{mrkt}.[8] Table 4 lists all parameter values used in the simulation exercise.

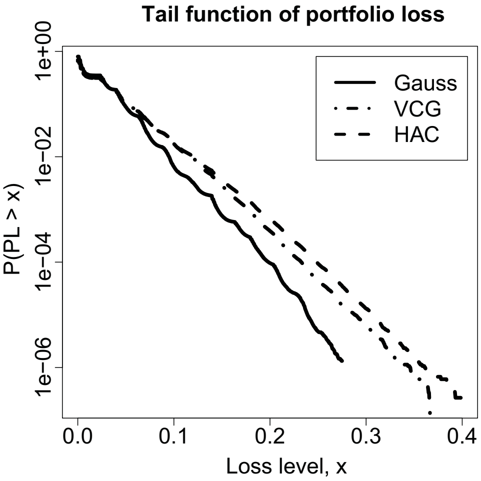
3.4. Simulation results

In this subsection I report results on a Monte Carlo simulation study for two test portfolios under the VCG framework and compare them with the outcomes of the HAC and Gaussian models given in Puzanova (2011). In all simulations I generate $s = 1.5 \times 10^7$ realisations of the portfolio loss variable in order to achieve more precise results.[9]

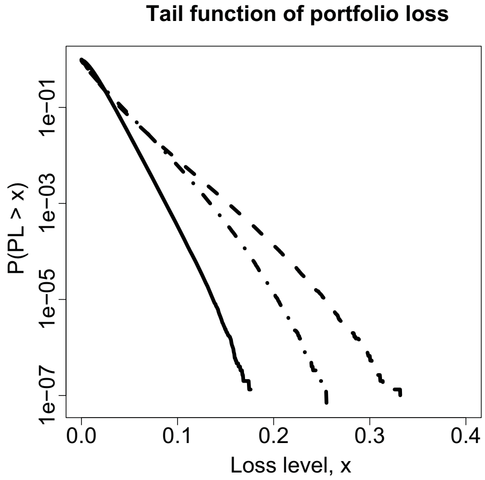
[8] I leave the issue of parameter estimation and further empirical investigations for future work.

[9] $s = 1.5 \times 10^7$ corresponds to the Monte Carlo estimation of the small probability $1 - q = 0.0001$ (which is equivalent to the estimation of $VaR_{0.9999}$ in terms of simulation efficiency) with an estimation error of at most 5% at the 95% confidence level.

Figure 9 demonstrates that the loss distribution based on the VCG has a heavier tail that the loss distribtion arising from non-tail-dependent Gaussian risk factors. For the same dependence parameters $\kappa_{(\cdot)}$, the HAC model leads to even more probability mass in the tail of the portfolio loss distribution due to the stronger lower-tail dependence of the underlying asset returns. The difference between three models is more pronounced in the case of the larger portfolio because the more debtors are in the portfolio, the more combinations of joint defaults are possible and the greater effect takes the tail dependence.



(a) Smaller portfolio



(b) Larger portfolio

Figure 9: Log-lin graphs of the simulated portfolio loss tail function for different model settings: Gaussian, Variance Compound Gamma (VCG) and Hierarchical Archimedean Copula (HAC). Results are given for two test portfolios containing 100 and 1,000 credit exposures.

As for the measures of the tail portfolio risk, the simulation results on VaR and ES at different levels q are presented in Table 5. In all cases the tail risk figures within the Gaussian setting are the lowest and those within the HAC setting are the highest, with the VCG figures lying inbetween. The difference becomes more distinct the further we go in the tail. Again, the risk of VaR/ES underestimation, if the assumption of the exponentially light-tailed distribution of asset returns and the linear dependence structure is wrong, is higher for the larger portfolio. For instance, under the VCG setting the maximum value of the smaller credit portfolio being at risk with a probability of 99.9% and portfolio loss expected if this VaR threshold is breached are around 25% higher than under the Gaussian setting. This figure rises to around 50% for the larger portfolio.

The results presented above demonstrate that, for the same linear correlation of asset returns, the model risk can be very considerable if the true distributions are skewed, heavy-tailed or/and exhibit lower-tail dependence.

The tail dependence properties of the underlying joint distribution of asset returns influence

Table 5: Comparison of VaR and ES for different settings

q	$\widehat{VaR}_q$ Gauss	VCG	HAC	$\widehat{ES}_q$ Gauss	VCG	HAC
Smaller portfolio						
0.9900	0.0955	0.1180	0.1210	0.1221	0.1433	0.1514
0.9950	0.1055	0.1355	0.1415	0.1335	0.1593	0.1712
0.9990	0.1455	0.1785	0.1875	0.1634	0.2030	0.2129
0.9995	0.1665	0.1930	0.2080	0.1921	0.2155	0.2330
0.9999	0.1985	0.2330	0.2485	0.2176	0.2582	0.2725
Larger portfolio						
0.9900	0.0615	0.0905	0.0950	0.0734	0.1102	0.1214
0.9950	0.0695	0.1045	0.1125	0.0814	0.1248	0.1386
0.9990	0.0880	0.1340	0.1530	0.1010	0.1506	0.1781
0.9995	0.0960	0.1465	0.1695	0.1105	0.1650	0.1930
0.9999	0.1135	0.1725	0.2065	0.1256	0.1897	0.2269

Note: VaR and ES at different levels q estimated by simulation for various parameter settings and three different models: Gaussian, Variance Compound Gamma (VCG) and Hierarchical Archimedean Copula (HAC). Results are given for two test portfolios containing 100 and 1,000 credit exposures.

to a great degree the tail behavior of the portfolio loss distribution. To illustrate the sensitivity of portfolio tail risk to the skewness and kurtosis parameters of the hierarchial VCG model, I report in Table 6 the VaR for the smaller portfolio for different parameter values. Here, a simplified model setting is considered with $\kappa_{Y_j} = \kappa_Y$ and $\mu_j = \mu$ for $j = 1, 2$. The simulation results demonstrate the impact of the increasing (absolute) parameter values on the portfolio VaR.

Finally, I would like to touch on the issue of simulation efficiency for large portfolio losses. For the Monte Carlo simulation carried out in accordance with the sampling algorithm in subsections 2.3, the CPU time needed for 1.5×10^7 simulation runs on the reference computer amounted to 21 min. (1.85 hours) for the VCG model in the case of the smaller (larger) portfolio. The long run times could be unacceptable for those practitioners who have to carry out many computations for large portfolios. Reducing the number of simulation runs would only shorten the computation time at the expense of precision. To avoid such an unfavorable trade-off, I recommend always bearing in mind that Importance Sampling (IS) or another variance reducing technique can be implemented. In the next section I derive a promising IS algorithm for the VCG portfolio model.

Table 6: Parameter sensitivity of the portfolio tail loss

		$\widehat{VaR}_{0.99}$			$\widehat{VaR}_{0.999}$		
$\kappa_{Z^{mrkt}}$	μ	$\kappa_Y = 0.2$	$\kappa_Y = 0.5$	$\kappa_Y = 0.9$	$\kappa_Y = 0.2$	$\kappa_Y = 0.5$	$\kappa_Y = 0.9$
				VCG model			
0.01	-0.5	0.1055	0.1315	0.1575	0.1670	0.1975	0.2380
	-0.7	0.1180	0.1540	0.1950	0.1775	0.2355	0.2985
	-0.9	0.1300	0.1825	0.2590	0.2010	0.2825	0.3485
0.05	-0.5	0.1090	0.1325	0.1600	0.1730	0.2040	0.2465
	-0.7	0.1245	0.1575	0.1960	0.1915	0.2475	0.3035
	-0.9	0.1350	0.1860	0.2555	0.2110	0.2840	0.3500
0.10	-0.5	0.1165	0.1380	0.1605	0.1765	0.2120	0.2520
	-0.7	0.1300	0.1615	0.2000	0.2070	0.2515	0.3080
	-0.9	0.1455	0.1960	0.2660	0.2245	0.2980	0.3500

Note: Parameter sensitivity of the VaR at different levels q with respect to the parameters of the Variance Compound Gamma model (VCG). Only the small portfolio containing 100 credit exposures is considered.

4. Importance Sampling algorithm

For the portfolio loss function within a Gaussian framework, a two-stage IS algorithm was introduced by Glasserman and Li (2005). Inspired by that paper, and using results from Kang and Shahabuddin (2005) and Merino and Nyfeler (2004), I work out a three-stage IS algorithm for the hierarchical VCG model based on the exponential tilting of the systematic factors and conditional portfolio loss distribution.

I begin with the transformation of the conditional portfolio loss distribution. The basic idea of the IS in this case is to shift the mean of the conditional loss distribution into the tail so that large losses would not be rare any more and VaR/ES could be estimated more efficiently. To do so the number of default events should be increased in a meaningful way by scaling up conditional PDs.

The simulation approach I used in the previous section was to sample the VCG asset returns and to compute the default indicators (3.4). Alternatively, it is possible to calculate individual PDs *conditional* on the specific realisation of the systematic factors (denoted by p)

$$p_{ji} = \Phi\left(\frac{F^{-1}_{\tilde{R}_{ji}}(PD_{ji}) + \mu_{ji} - \mu_{ji}\cdot(Z_j \mid Z^{mrkt})}{\sqrt{(Z_j \mid Z^{mrkt})\cdot\left[1-\mu^2_{ji}(\kappa_{Z^{mrkt}}+\kappa_{Y_j})\right]}}\right) \tag{4.1}$$

and to sample default indicators D_{ji} from mutually independent Bernoulli distributions with parameters p_{ji}.

As described in great detail in the literature mentioned at the beginning of this section, the exponential twisting of the conditional PDs will lead to the desirable transformation of the conditional loss distribution. The exponentially twisted conditional PDs are given as:

$$p^*_{ji}(\theta) := \frac{e^{L_{ji}\theta} p_{ji}}{1 - p_{ji} + e^{L_{ji}\theta} p_{ji}}, \tag{4.2}$$

where θ is a common twisting parameter which can be uniquely identified, as will be described later.

Sampling default indicators $D_{ji} \sim Be\big(p^*_{ji}(\theta)\big)$ biases the simulated portfolio loss distribution. The bias can be corrected, however, by means of the likelihood ratio, which is the ratio of the original and the transformed conditional loss distributions. This likelihood ratio for the portfolio loss function conditional on the systematic factors $\mathbf{Z} = (Z_1, \cdots, Z_m)$ can be given as follows:

$$L(PL \mid \mathbf{Z}) = \exp\left(-\theta PL + \sum_{j=1}^{m} \sum_{i=1}^{n_j} \ln\Big(1 - p_{ji} + e^{L_{ji}\theta} p_{ji}\Big)\right). \tag{4.3}$$

The second term in brackets represents the cumulant generating function of the conditional loss distribution which I denote by $C_{PL|\mathbf{Z}}(\theta)$ in the following. The first derivative of this term equals the mean of the distribution. Thus, to shift the mean into the tail, I have to choose the twisting parameter θ such that the mean equals the desirable quantile x_q:

$$\theta_{x_q} := \Big\{\theta \,:\, \big[C_{PL|\mathbf{Z}}(\theta)\big]' = x_q\Big\}. \tag{4.4}$$

It is important to point out that, for practical purposes, there is no need to solve (4.4) repeatedly for different values of x_q. It is sufficient to choose one single value of x_q far in the tail but less than VaR_q. x_q can be chosen, for instance, based on a quick preliminary Monte Carlo simulation. Its exact value does not considerably affect the simulation efficiency.

Let us now go on with the exponential twisting of the Gamma distributed systematic factors. Consider a Gamma variable with the shape parameter β and rate parameter λ. The corresponding exponentially twisted pdf at point x arises as a ratio of the original Gamma pdf and its moment generating function for a $\vartheta < \lambda$ multiplied by $e^{\vartheta x}$:

$$f_*(x;\vartheta) = \frac{e^{\vartheta x}}{(1-\vartheta/\lambda)^{-\beta}} \cdot \frac{e^{-\lambda x}}{\Gamma(\beta)\lambda^{-\beta}} \cdot x^{\beta-1} = \frac{e^{-\lambda^*(\vartheta)x}}{\Gamma(\beta)\big(\lambda^*(\vartheta)\big)^{-\beta}} \cdot x^{\beta-1},$$

which turns out to be a Gamma pdf with the parameters β and $\lambda^*(\vartheta) := \lambda - \vartheta$.

According to the above result, I use the Gamma distribution with the parameters $(1/\kappa_{Z^{mrkt}}$, $1/\kappa_{Z^{mrkt}} - \vartheta)$ to sample the market-level factor Z^{mkrt}. As for the sector-specific systematic factors, these factors are independently Gamma-distributed conditional on a realisation of the market-level factor $Z^{mkrt} = z^{mkrt}$. Thus, I sample the sector-specific factors from mutually independent Gamma distributions with parameters $\left(z^{mkrt}/\kappa_{Y_j},\, 1/\kappa_{Y_j} - \vartheta_j\right)$. I will show in the following how to choose the twisting parameters ϑ and ϑ_j. But first, let us consider the likelihood ratios needed to correct the bias of sampling from the transformed distributions.

The likelihood ratio is always the ratio of the original distribution to the sampling distribution. For the twisted pdf of Z^{mrkt} it can be written as:

$$L(Z^{mrkt}) = \exp\left(-\vartheta Z^{mrkt} - \frac{1}{\kappa_{Z^{mrkt}}}\ln(1-\kappa_{Z^{mrkt}}\vartheta)\right), \tag{4.5}$$

where the second term in brackets is the cumulant generating function of Z^{mrkt}, which I denote by $C_{Z^{mrkt}}(\vartheta)$. For the twisted conditional variables $Z_j \mid Z^{mrkt}$, the likelihood ratio reads:

$$L(\mathbf{Z} \mid Z^{mrkt}) = \exp\left(-\sum_{j=1}^{m}\vartheta_j Z_j \mid Z^{mrkt} - \sum_{j=1}^{m}\frac{Z^{mrkt}}{\kappa_{Y_j}}\ln(1-\kappa_{Y_j}\vartheta_j)\right). \tag{4.6}$$

I denote the sum of the cumulant generating functions (the second term in brackets) by $\sum_{j=1}^{m} C_{Z_j|Z^{mrkt}}(\vartheta_j)$. The overall likelihood ratio is just the product of (4.3), (4.5) and (4.6):

$$\begin{aligned} L(PL) = \exp\Bigg(& -\theta PL - \vartheta Z^{mrkt} - \sum_{j=1}^{m}\vartheta_j Z_j \mid Z^{mrkt} \\ & + C_{PL|\mathbf{Z}}(\theta) + C_{Z^{mrkt}}(\vartheta) + \sum_{j=1}^{m} C_{Z_j|Z^{mrkt}}(\vartheta_j)\Bigg). \end{aligned} \tag{4.7}$$

In order to choose appropriate values for the twisting parameters ϑ and ϑ_j, I adopt theorem 1 in Bassamboo and Jain (2006, p. 743) for an asymptotically optimal IS algorithm. According to this theorem, and keeping in mind that conditional PDs are stochastically increasing in the systematic factors, I can set optimal twisting parameters ϑ^* and ϑ_j^* by solving the following optimisation problem:

$$\begin{aligned} \sup_{z^{mrkt};z_1,\ldots,z_m\in\mathbb{R}_+} \Bigg[& \inf_{\vartheta\in(0,1/\kappa_{Z^{mrkt}})}\left(C_{Z^{mrkt}}(\vartheta) - \vartheta z^{mrkt}\right) \\ & + \sum_{j=1}^{m}\inf_{\vartheta_j\in(0,1/\kappa_{Y_j})}\left(C_{Z_j|Z^{mrkt}}(\vartheta_j) - \vartheta_j z_j\right) + \inf_{\theta\in\mathbb{R}_+}\left(C_{PL|\mathbf{Z}}(\theta) - \theta x_q\right)\Bigg]. \end{aligned} \tag{4.8}$$

In words, I maximise simultaneously (i) the probability that realisations of the systematic factors Z^{mrkt} and $Z_j \mid Z^{mrkt}$ are greater than certain values and (ii) the probability that portfolio loss is greater that a desirable quantile.

Summarising, I propose the following IS algorithm for the simulation of the portfolio loss distribution within the VCG setting:

Simulation algorithm 2: Importance Sampling for the VCG framework

- Solve (4.8) to define parameters ϑ^* and ϑ_j^*, $j = 1, \dots, m$.
- Repeat the following simulation steps s times:
  - Sample $Z^{mrkt} \sim \Gamma\big(1/\kappa_{Z^{mrkt}}, 1/\kappa_{Z^{mrkt}} - \vartheta^*\big)$.
  - Sample $Z_j \mid Z^{mrkt}$, $j = 1, \dots, m$ from independent Gamma distributions with parameters $\big(Z^{mrkt}/\kappa_{Y_j}, 1/\kappa_{Y_j} - \vartheta_j\big)$.
  - Compute conditional PDs p_{ji}, $i = 1, \dots, n_j$, $j = 1, \dots, m$ as in (4.1).
  - If $x_q \le E[PL \mid \mathbf{Z}] = \sum_{j=1}^m \sum_{i=1}^{n_j} L_{ji} \cdot p_{ji}$, set $\theta_{x_q} = 0$. Otherwise define θ_{x_q} by solving (4.4).
  - Compute twisted conditional PDs $p_{ji}^*(\theta_{x_q})$ according to (4.2).
  - Sample D_{ji} from independent Bernoulli distributions with parameters $p_{ji}^*(\theta_{x_q})$.
  - Compute portfolio loss rate $PL^k = \sum_{j=1}^m \sum_{i=1}^{n_j} L_{ji} \cdot D_{ji}$ for the kth simulation run.
  - Compute the corresponding likelihood ratio $L(PL^k)$ according to (4.7), thereby set $\theta := \theta_{x_q}$, $\vartheta_{ji} := \vartheta_{ji}^*$ and $\vartheta := \vartheta^*$.
- Compute IS estimators for the portfolio loss rate distribution, VaR and ES as follows:

$$\hat{F}_{PL}^{IS}(x_q) \equiv 1 - \hat{q}^{IS} = 1 - \frac{1}{s}\sum_{k=1}^{s} L(PL^k)\mathbb{1}_{(x_q,1]}(PL^k),$$

$$\widehat{VaR}_q^{IS}(PL) = \inf\big\{x \in [0,1] \,:\, \hat{F}_{PL}^{IS}(x) \ge q\big\},$$

$$\widehat{ES}_q^{IS}(PL) = \frac{\sum_{k=1}^{s} PL^k \cdot L(PL^k)\,\mathbb{1}_{\left(\widehat{VaR}_q^{IS}(PL),1\right]}(PL^k)}{\sum_{k=1}^{s} L(PL^k)\,\mathbb{1}_{\left(\widehat{VaR}_q^{IS}(PL),1\right]}(PL^k)}$$
$$+\,\widehat{VaR}_q^{IS}(PL)\frac{1 - \hat{q}^{IS} - \frac{1}{s}\sum_{k=1}^{s} L(PL^k)\,\mathbb{1}_{\left(\widehat{VaR}_q^{IS}(PL),1\right]}(PL^k)}{1 - \hat{q}^{IS}}.$$

For the sake of completeness, I should mention that the complete IS procedure would be the following: (i) solving (4.8) for the beginning of the simulation; (ii) sampling Z^{mrkt} accordingly; (iii) solving

$$\sup_{z_1,\dots,z_m \in \mathbb{R}_+} \left[\sum_{j=1}^{m} \inf_{\vartheta_j \in (0,1/\kappa_{Y_j})} \big(C_{Z_j|Z^{mrkt}}(\vartheta_j) - \vartheta_j z_j\big) + \inf_{\theta \in \mathbb{R}_+} \big(C_{PL|\mathbf{Z}}(\theta) - \theta x_q\big)\right] \tag{4.9}$$

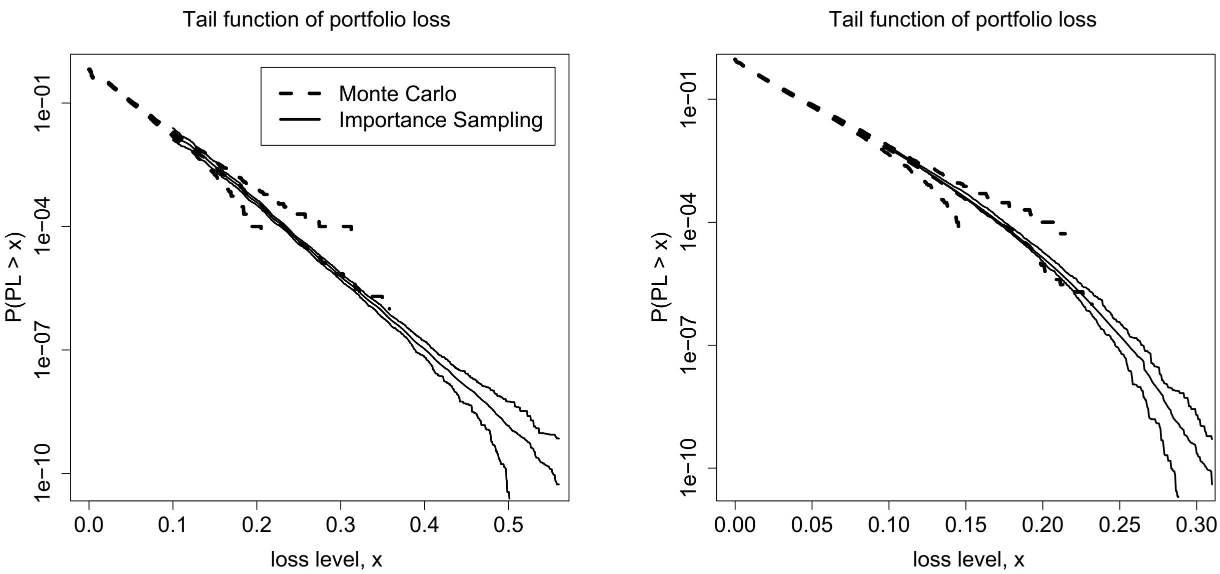
to set ϑ_j^* conditionally on Z^{mrkt}; (iv) sampling $Z_j \mid Z^{mrkt}$ accordingly; (v) solving (4.4) to set θ_{x_q} conditional on $Z_j \mid Z^{mrkt}$; (vi) computing conditional PDs and sampling default indicators accordingly. However, solving (4.9) for a particular realisation of Z^{mrkt} can be skipped in order to speed up the simulation without any material loss of efficiency. Thus, it is only necessary to solve (4.8) for ϑ^* and ϑ_j^* just once before beginning the simulation.

The simulation algorithm 2 leads to a considerable variance reduction of the estimated portfolio quantiles, ensuring stable results for VaR and ES. Table 7 clarifies this statement by an example. For both portfolios, I repeat the IS and Monte Carlo simulation scenarios 100 times in order to compute the respective sample variances of the VaR estimates. The results show that, even for as few as 10^3 simulation runs per scenario, the variation of the IS estimates for VaR at different levels q is 1 to 3 orders of magnitude smaller than the variation of Monte Carlo estimates based on 10^4 simulation runs, the CPU time elapsed being fairly comparable. It is noteworthy that IS delivers stable results for portfolio losses far in the tail where Monte Carlo simply fails to generate any sufficient number of realisations because, on average, it only generates $(1-q) \times s$ outcomes lying beyond the q-quantile of the loss distribution: see Figure 10 for illustration. Overall, I judge the gain in simulation efficiency when using the IS algorithm as well worth the time needed to implement it into a suitable programming language.

Table 7: The variance reduction factor for VaR obtained via IS

	Variance reduction factor	
q	smaller portfolio	larger portfolio
0.995	1.16	4.17
0.997	2.23	7.62
0.999	10.49	12.96
0.9995	25.87	15.60
0.9997	37.45	24.03
0.9999	140.66	54.54
0.99995	262.43	111.94
0.99997	375.51	129.60
CPU time: MC	1.62 min.	9.13 min.
CPU time: IS	1.73 min.	9.79 min.

Note: For Variance Compound Gamma settings the variance reduction factor for VaR estimation at different levels q is given. It is a ratio of the sample variance of VaR estimated via Monte Carlo (MC) to the sample variance of VaR estimated via Importance Sampling (IS). The sample variances were computed on the basis of 100 independent scenarios each containing 10^4 simulation runs for MC and 10^3 simulation runs for IS. Also given is the CPU time elapsed. The results are presented for two test portfolios containing 100 and 1,000 credit exposures.



(a) smaller portfolio

(b) larger portfolio

Figure 10: Comparison of Monte Carlo and Importance Sampling simulation results with 10^4 and 10^3 replications respectively. In the log-lin graphs of the tail function of the portfolio loss rate the pointwise means and 95% confidence intervals calculated from 100 independent scenarios are given.

5. Conclusions

The Variance Compound Gamma (VCG) model introduced in this paper can be useful for modelling asset returns both in univariate and multivariate settings. The underlying VCG process is a pure-jump Lévy process that arises from a two-stage Gamma-time change of a Brownian motion.

The *univariate* VCG process can be useful for the univariate modelling of stock returns, for the purposes of option or credit derivative pricing. This stochastic pure-jump process satisfies such important stylised facts of stock returns as asymmetry and excess kurtosis. Moreover, it overcomes the major shortcoming of Black-Scholes-type Gaussian models with continuous sample paths of asset returns and allows for unanticipated default events triggered by sudden shocks in the asset price. This last feature is crucial to the modelling of credit risk.

The *multivariate* VCG framework has several advantages in terms of modelling credit portfolios. The specific time-change procedure proposed in this paper generates a hierarchical dependence structure that allows for a stronger dependence within specified sectors or sub-portfolios and for a weaker dependence between them. The model is flexible enough for a differentiated treatment of sub-portfolios with respect to their tail dependence properties. It is worth noting that, although the copula function underlying the one-period, static VCG model cannot be given explicitly, a closed, copula-like representation of the joint characteristic function of asset returns exists. It is, in fact, the nested Archimedean copula, introduced in Puzanova (2011) and applied to marginal characteristic functions.

This paper shows that the VCG copula family, which joins asset returns of firms operating in one particular sector (or attributed to one particular sub-portfolio), is ordered with respect to each of its parameters. This result implies that higher absolute values of the skewness parameters of asset returns and/or of the variance parameters of the underlying stochastic times lead to the stronger dependence as given by the concordance measure known as Kendall's tau. The magnitude of tail dependence is also increasing in skewness parameters. By contrast, the grouped VCG copula, which joints asset returns of firms operating in different sectors (or attributed to different sub-portfolios), does not exhibit tail dependence. This copula family is also ordered but has lower Kendall's tau for comparable parameter values.

From a computational point of view the advantage of the VCG model is that simulation can be easily accomplished using pseudo-random number generators for normal and Gamma distributions which are standard components of mathematical and statistical packages. The variance-reducing Importance Sampling algorithm provided in this paper increases simulation efficiency considerably.

From the perspective of the portfolio credit risk assessment, the main advantage of the multivariate VCG model over a Gaussian framework is that the stochastic time change applied gives rise to tail dependence of asset returns and, in turn, to clustering default events. Therefore, implementation of the suggested model could have far-reaching implications for risk controlling and banking regulation and, on a large scale, for financial stability. It would result in a more conservative assessment of portfolio credit risk and, consequently, higher capital requirements. Therefore, the model is able to counter the systematic underestimation of credit risk in banking sector – one of the underlying causes of the recent financial turmoil.

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A. The VCG probability distribution function

The pdf of VCG-distributed asset returns can only be given in its integral representation. Bearing in mind that

- conditionally on a realisation z_j of the sector-specific business time, the asset return of each single firm in sector j is normally distributed and
- conditionally on a realisation z^{mrkt} of the market business time, the sector-specific subordinators are Gamma-distributed,

the VCG pdf can be given as:

$$
\begin{aligned}
f_{R_{ji}(t)}(x) = & \int_0^\infty \int_0^\infty \frac{1}{\sigma_{ji}\sqrt{2\pi z_j}} \exp\left(-\frac{(x-\mu_{ji}z_j)^2}{2\sigma_{ji}^2\, z_j}\right) \\
& \times \frac{e^{-z_j/\kappa_{Y_j}}}{\Gamma(z^{mrkt}/\kappa_{Y_j})\kappa_{Y_j}^{z^{mrkt}/\kappa_{Y_j}}} z_j^{z^{mrkt}/\kappa_{Y_j}-1}\, dz_j \\
& \times \frac{e^{-z^{mrkt}/\kappa_{Z^{mrkt}}}}{\Gamma(t/\kappa_{Z^{mrkt}})\kappa_{Z^{mrkt}}^{t/\kappa_{Z^{mrkt}}}} (z^{mrkt})^{t/\kappa_{Z^{mrkt}}-1}\, dz^{mrkt} \\
= & \frac{2}{\sigma_{ji}\sqrt{2\pi}\,\Gamma(t/\kappa_{Z^{mrkt}})\kappa_{Z^{mrkt}}^{t/\kappa_{Z^{mrkt}}}} \exp\left(\frac{x\,\mu_{ji}}{\sigma_{ji}^2}\right) \\
& \times \int_0^\infty \frac{e^{-z^{mrkt}/\kappa_{Z^{mrkt}}}\cdot (z^{mrkt})^{t/\kappa_{Z^{mrkt}}-1}}{\Gamma(z^{mrkt}/\kappa_{Y_j})\kappa_{Y_j}^{z^{mrkt}/\kappa_{Y_j}}} \left(\frac{|x|/\sigma_{ji}}{\sqrt{\mu_{ji}^2/\sigma_{ji}^2+2/\kappa_{Y_j}}}\right)^{z^{mrkt}/\kappa_{Y_j}-1/2} \\
& \times K_{(z^{mrkt}/\kappa_{Y_j}-1/2)}\left(\frac{|x|}{\sigma_{ji}}\sqrt{\frac{\mu_{ji}^2}{\sigma_{ji}^2}+\frac{2}{\kappa_{Y_j}}}\right) dz^{mrkt}.
\end{aligned}
$$

$\Gamma(\cdot)$ is the gamma function and $K_{(\lambda)}(\cdot)$ denotes the modified Bessel function of the third kind with the index λ:

$$
K_{(\lambda)}(x) = \frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left[-\frac{x}{2}(y+y^{-1})\right] dy, \quad x>0.
$$

Mean, variance, skewness and excess kurtosis of the VCG-distributed asset returns are as follows:

$$
E\big[R_{ji}(t)\big] = \mu_{ji}t, \tag{A.1}
$$

$$
\operatorname{var}\big(R_{ji}(t)\big) = \Big[\sigma_{ji}^2 + \mu_{ji}^2\big(\kappa_{Z^{mrkt}}+\kappa_{Y_j}\big)\Big]\, t, \tag{A.2}
$$

$$\gamma_1\big(R_{ji}(t)\big) = \mu_{ji}\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)\frac{3\sigma_{ji}^2 + 2\mu_{ji}^2\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)}{\sqrt{t}\Big[\sigma_{ji}^2 + \mu_{ji}^2\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)\Big]^{3/2}}$$

$$- \frac{2\mu_{ji}^3\kappa_{Z^{mrkt}}\kappa_{Y_j}}{\sqrt{t}\Big[\sigma_{ji}^2 + \mu_{ji}^2\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)\Big]^{3/2}},$$

$$\gamma_2\big(R_{ji}(t)\big) = \frac{3\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)}{t}\left(2 - \frac{\sigma_{ji}^4}{\big[\sigma_{ji}^2 + \mu_{ji}^2\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)\big]^2}\right)$$

$$- \frac{\mu_{ji}^2\,\kappa_{Z^{mrkt}}\kappa_{Y_j}\Big[6\,\sigma_{ji}^2 + \mu_{ji}^2\big(6\kappa_{Z^{mrkt}} + 7\kappa_{Y_j}\big)\Big]}{t\big[\sigma_{ji}^2 + \mu_{ji}^2\big(\kappa_{Z^{mrkt}} + \kappa_{Y_j}\big)\big]^2}.$$

The integral representation of the multivariate pdf of VCG-distributed asset returns can be given as follows (for $t = 1$, so the subscript t has been dropped):

$$f_{\mathbf{R}(t)}(\mathbf{x}) = \frac{2^m}{(2\pi)^{n/2}\,|\Sigma|^{1/2}\,\Gamma(t/\kappa_{Z^{mrkt}})\kappa_{Z^{mrkt}}^{t/\kappa_{Z^{mrkt}}}}\exp\Big(\mathbf{x}'\Sigma^{-1}\boldsymbol{\mu}\Big)$$

$$\times \int_0^\infty \prod_{j=1}^m \frac{e^{-z^{mrkt}/\kappa_{Z^{mrkt}}}\cdot\big(z^{mrkt}\big)^{t/\kappa_{Z^{mrkt}}-1}}{\Gamma(z^{mrkt}/\kappa_{Y_j})\kappa_{Y_j}^{z^{mrkt}/\kappa_{Y_j}}}\left(\frac{\mathbf{Q}_j}{\sqrt{\boldsymbol{\mu}_j'\Sigma_j^{-1}\boldsymbol{\mu}_j + 2/\kappa_{Y_j}}}\right)^{z^{mrkt}/\kappa_{Y_j}-n_j/2}$$

$$\times K_{(z^{mrkt}/\kappa_{Y_j}-n_j/2)}\left(\mathbf{Q}_j\cdot\sqrt{\boldsymbol{\mu}_j'\Sigma_j^{-1}\boldsymbol{\mu}_j + \frac{2}{\kappa_{Y_j}}}\right)\,dz^{mrkt}.$$

In the expression above I use $\mathbf{x}$, $\boldsymbol{\mu}$ and Σ to denote the vector of arguments of the distribution function, the vector of skewness parameters and the variance matrix of the Gaussian part respectively (all ordered with respect to sectors). The sub-vectors $\mathbf{x}_j$ and $\boldsymbol{\mu}_j$ and the submatrix Σ_j represent the corresponding parameters for sector j. $\mathbf{Q}_j = \sqrt{\mathbf{x}_j'\Sigma_j^{-1}\mathbf{x}_j}$ is the Mahalanobis distance between the elements of $\mathbf{x}_j$.

B. HAC representation for the VCG framework

Starting from expression (2.17), which is based on the marginal *conditional* cfs, I derive in this appendix a copula representation of the joint cf of VCG-distributed asset returns, which is a function of *unconditional* marginal cfs:

$$
\begin{aligned}
&E_{Z^{mrkt}}\left[\prod_{j=1}^{m} E_{Z_j|Z^{mrkt}}\left[\prod_{i=1}^{n_j}\phi_{R_{ji}|Z_j}(\theta_{ji})\right]\right] \qquad \text{(B.1)}\\
&= E_{Z^{mrkt}}\left[\prod_{j=1}^{m} E_{Z_j|Z^{mrkt}}\left[\exp\Big(-Z_j \mid Z^{mrkt}\sum_{i=1}^{n_j}\varphi_{Z_j}^{-1}\circ\phi_{R_{ji}}(\theta_{ji})\Big)\right]\right]\\
&= E_{Z^{mrkt}}\left[\prod_{j=1}^{m}\varphi_{Y_j}^{Z^{mrkt}}\Big(\sum_{i=1}^{n_j}\varphi_{Z_j}^{-1}\circ\phi_{R_{ji}}(\theta_{ji})\Big)\right]\\
&= E_{Z^{mrkt}}\left[\exp\Big\{-Z^{mrkt}\sum_{j=1}^{m}\varphi_{Z^{mrkt}}^{-1}\circ\varphi_{Z_j}\Big(\sum_{i=1}^{n_j}\varphi_{Z_j}^{-1}\circ\phi_{R_{ji}}(\theta_{ji})\Big)\Big\}\right]\\
&= \varphi_{Z^{mrkt}}\left[\sum_{j=1}^{m}\varphi_{Z^{mrkt}}^{-1}\circ\varphi_{Z_j}\Big(\sum_{i=1}^{n_j}\varphi_{Z_j}^{-1}\circ\phi_{R_{ji}}(\theta_{ji})\Big)\right].
\end{aligned}
$$

The aim of the transformation, accomplished after the first equality sign is to proceed to a representation which incorporates unconditional marginal cfs of R_{ji}. To do so, I use the expression (2.12) that implies $\psi_{X_{ji}}(\theta) = -\varphi_{Z_j(t)}^{-1}\circ\phi_{R_{ji}}(\theta)$ by inversion. Conditionally on $Z_j \mid Z^{mrkt}$, the normally distributed idiosyncratic part of asset returns in (2.14) is the only random part. Taken together, these statements imply the following chain of transformations:

$$
\begin{aligned}
\phi_{R_{ji}|Z_j}(\theta) &= E\left[\exp\Big(i\theta\mu_{ji}\, Z_j \mid Z^{mrkt} + i\theta\sigma_{ji}\sqrt{Z_j \mid Z^{mrkt}}W_{ji}\Big)\right]\\
&= \exp\big(i\theta\mu_{ji}\, Z_j \mid Z^{mrkt} - \tfrac{1}{2}\theta^2\sigma_{ji}^2 Z_j \mid Z^{mrkt}\big)\\
&= \exp\big(Z_j \mid Z^{mrkt}\cdot\psi_{X_{ji}}(\theta)\big)\\
&= \exp\big(-Z_j \mid Z^{mrkt}\cdot\varphi_{Z_j}^{-1}\circ\phi_{R_{ji}}(\theta)\big)
\end{aligned}
$$

Now, expression in form $E_{Z_j|Z^{mrkt}}\big[\exp\big(-Z_j \mid Z^{mrkt}\theta_{ji}\big)\big]$ represents the LT of the conditional random variable $Z_j \mid Z^{mrkt}$, which is Gamma distributed. This LT equals the LT of the Gamma random variable Y_j to the power of Z^{mrkt}, which leads to the expression given after the second equality sign in (B.1).

Solving equation (2.9) for φ_{Y_j} and substituting $\varphi_{Y_j}^{Z^{mrkt}}$ by the resulting expression gives the representation after the fourth equality sign in (B.1).

Replacing the remaining expectation operator with the LT of Z^{mrkt} delivers the final line in (B.1). From this last expression the HAC representation (2.26) follows.

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